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Anleitung Nr:

Shot noise

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1 Shot noise [1]

Certain physical processes within a conductor cause noise to appear in the current flow through the conductor. Some of these processes are inevitably related to the transport of electronic charges such that they cannot be avoided in principle. While the discontinuous emission of photons was investigated by Campbell in 1909 [2], first attempts to understand electrical current noise were made in 1918 by Walter Schottky on vacuum tubes [3]. In this experiment you will use the vacuum tube depicted in Fig. 1 for generating noise. An examplary of such vacuum tube is also present on the table of the VP experiment.

Figure 1: The vacuum tube used in this experiment for generating noise.

Schottky distinguished two types of noise:

- *thermal noise.* This contribution is often also called *Johnson–Nyquist noise*, after the experimentalist M.B. Johnson and the theoretician H. Nyquist, who studied thermal noise in detail [4, 5]. Today we know that thermal noise is present in all electronic conductors at finite temperature. It does not require a finite mean current to flow through the conductor, but appears as current noise as soon as the two sides of the conductor are connected. It is therefore an equilibrium phenomenon.

- *shot noise.* Shot noise (German: Schrotrauschen) arises because electronic charge is transported in quantized portions. Typically these portions have the size of the elementary charge $|e|$, but a few notable exceptions exist, e.g., Cooper pairs in superconductors ($2|e|$), or fractionally charged quasiparticles in the fractional quantum Hall effect (e.g., $|e|/3$). Shot noise arises only if a finite mean current is driven through a conductor and is therefore a nonequilibrium phenomenon. Shot noise does not occur in all conductors. For example, in macroscopic metallic conductors shot noise is suppressed because the sample size exceeds the inelastic electronic mean free path by orders of magnitude. Individual segments of the material fluctuate independently and the mean noise amplitude is strongly reduced by averaging. In systems, where the electronic mean free path is comparable to the extent of the system, shot noise is of importance.

At the time of Walter Schottky, shot noise could be observed and studied in vacuum tubes. Shot noise can also be observed in vacuum photodiodes [6, 7]. In fundamental research today, shot noise gives important insights about the correlated motion of electrons, which is due to the Pauli exclusion principle or electron–electron interactions, in all kinds of nanoscale conductors, such as quantum wires, quantum dots [1], and superconducting Josephson junctions. It also plays a role in lasers.

Usually, in electronic transport measurements a constant voltage V is applied to the device under investigation, and a current $I(t)$ is measured. The time averaged current $\langle I \rangle$ (also called *dc*, meaning <u>d</u>irect <u>c</u>urrent) is

$$\langle I \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt I(t). \tag{1}$$

Fluctuations of the current in time around this average

$$\Delta I(t) = I(t) - \langle I \rangle \tag{2}$$

are called the noise current, or simply the noise. The time averaged current noise is quantified by the mean square fluctuation amplitude

$$\langle \Delta I^2 \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt \Delta I^2(t). \tag{3}$$

Walter Schottky found the expression

$$\langle \Delta I^2 \rangle = 2|e| \langle I \rangle \Delta f, \tag{4}$$

for the magnitude of the shot noise, known today as the *Schottky formula*. In this equation, Δf is the bandwidth of the apparatus used to measure the noise. The quantity $S(\omega) = S_0 = 2|e|\langle I \rangle$ is the power spectral density of the shot noise current.

According to eq. (4), shot noise provides the opportunity to determine the elementary charge $|e|$ from a measurement of the average current and the noise. Already in 1924, Hull and Williams [8] used this approach with an error of $\pm 2\%$. In 1952, Stigmark determined $|e|$ with an error of $\pm 0.4\%$ [9]. With this study, the experimental accuracy limit of the determination of $|e|$ using shot noise had been reached.

In this experiment you will be measuring the shot noise generated by a vacuum tube. In such a tube individual electrons leave the heated cathode electrode by thermionic emission. Electrons are emitted randomly and independently. Each electron traveling ballistically from cathode to anode causes a current pulse. The sum of the random and independent current pulses creates the mean current and the noise. Using a resonant circuit as an impedance the noise current is transformed into a noise voltage [see Fig. 2(a)]. This noise voltage is then amplified and rectified. The rectification allows you to measure the magnitude of the time averaged shot noise. Proper calibration of the setup will allow you to determine $\langle \Delta I^2 \rangle$ and $\langle I \rangle$ giving access to the value of the elementary charge $|e|$.

2 Apparatus and Instrumentation

The apparatus consists of

- the noise generator including the vacuum tube and the resonant circuit,
- the voltage amplifier used to amplify the voltage noise,
- the rectifier used to rectify the noise signal into a dc voltage,
- an ac-voltmeter for measuring the rms (root mean square) amplitude of voltages.

You will also find auxiliary instrumentation for calibrating and analyzing the measurement system, namely,

- a high-frequency oscillator,
- an LRC-bridge,
- an oscilloscope.

The noise generator. A simplified diagram of the electrical circuit in the noise generator is shown in Fig. 2(a). The cathode voltage is kept constant close to -150 V. For dc signals, the anode is shorted to ground via the inductor L of the resonant circuit, allowing to keep the constant dc voltage between cathode and anode. For frequency components of the diode current $I(t)$ close to the resonant frequency of the LC-circuit (or tank-circuit), the impedance of this circuit is high, and the current $I(t)$ causes a significant voltage drop across it. This voltage drop can be measured as the output voltage V_{A}.

The output voltage of the noise generator is accessible via a single BNC-connector.[1] Terminals (A) and (B) in Fig. 2 refer to the inner and the outer conductor (shield) of a single BNC connector.

The diode itself and capacitive loads connected to the output of this circuit [e.g., coaxial cables, input capacitance of the voltage amplifier connected between (A) and (B)] contribute to the total capacitance C_{tot} of the resonant circuit in addition to C. Its resonance frequency is then given by

$$f_{\mathrm{res}} = \frac{1}{2\pi\sqrt{LC_{\mathrm{tot}}}}.$$

[1]The abbreviation BNC stands for Bayonet Neill-Concelman, as it has a bayonet mount, and it was invented by Paul Neill and Carl Concelman.

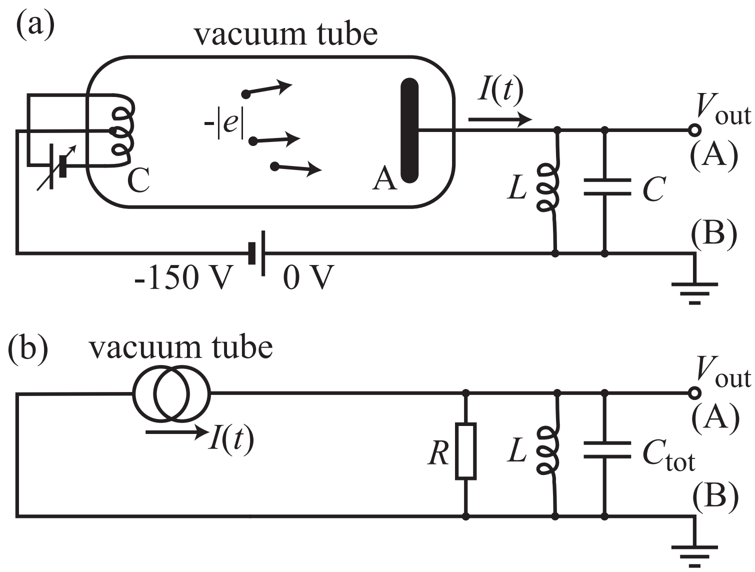


Figure 2: (a) Schematic circuit diagram of the noise generator. Within the vacuum diode, C denotes the cathode and A the anode. (b) Equivalent circuit of the noise generator.

In the same spirit, the resonant circuit will suffer from internal losses (e.g., dielectric losses in the capacitors, resistive losses in the coil, resistive losses in the tube) which determine the quality factor of the circuit and its bandwidth. As the resonant circuit shortens all frequencies outside its own bandwidth to ground, the bandwidth of the resonant circuit is the bandwidth Δf of the measurement (c.f. eq. (4)). The losses in the circuit can be represented in the equivalent circuit by a resistance R parallel to L and C [see Fig. 2(b)].

The noise diode is a directly heated vacuum tube (see Fig. 1).[2] Fig. 3(a) schematically shows the current–voltage characteristics of such a vacuum tube, while Fig. 3(b) shows the current–voltage characteristic for the tube used in this experiment for a particular heater current. In this experiment the tube is operated in the saturation region, where the current shows very little dependence on the applied cathode–anode voltage. The tube can therefore essentially be seen as a current source [see equivalent circuit in Fig. 2 (b)]. Its

[2]The tube used here is a Sylvania Type 5722 noise generating diode. Indirectly heated tubes contain oxide cathodes. They have the advantage that they can also be heated using alternating current. On the other hand, the directly heated tube we use, shows a higher dependence of the emission current on the heating power, which is desired in this experiment.

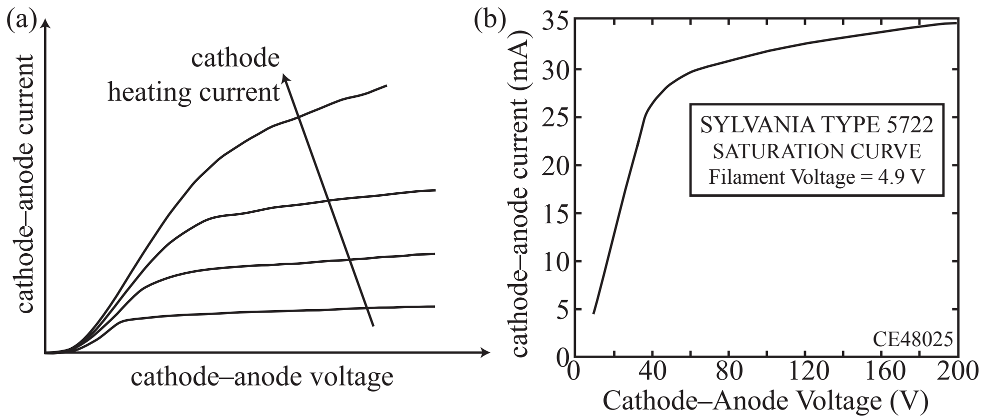


Figure 3: (a) Typical current–voltage characteristic of a vacuum diode for different heating currents. (b) Current–voltage characteristic of the vacuum diode used in this experiment (taken from the data sheet of the tube).

output current can be controlled via the heating current applied to the cathode. The dc diode current is kept constant at a tunable set point by a control loop adapting the cathode heating power.

The voltage amplifier. We use the voltage amplifier to measure the diode current and current noise which is converted into voltage and voltage noise by the resonant circuit. The amplifier consists of three stages, which are the same in principle. Two of them amplify the voltage by a factor of 20, the first stage has an adjustable amplification [10]. The input voltage is ac-coupled to the first amplifier stage via a 10 nF series capacitor located inside the amplifier casing. This ac-coupling makes sure that only the fluctuating part of the voltage couples into the amplifier, and any dc voltages are blocked. The input impedance of the amplifier is 1 MΩ. The frequency response of the amplifier is essentially flat in the frequency band between 20 kHz and 4.5 MHz.

Task: Determine experimentally the maximum output voltage of the amplifier. If the output voltage approaches this value from below, the amplifier becomes overdriven and the amplifier output voltage goes into saturation.

The rectifier. A multiplier squares the signal coming from the amplifier. The basic component is an integrated analog multiplier from Motorola. The time average is then performed by an RC low-pass filter.

Task:

1. Determine experimentally, which maximum signal you can apply at the two inputs of the multiplier (connected in parallel) in order to read full-scale output on the built-in ammeter.

2. Connect the coaxial output of the multiplier to the oscilloscope. How does the output frequency compare to the frequency of the input signal? Can you explain the observed behaviour?

3. Studying the circuit diagram of the multiplier, can you find out what the time-constant and the bandwidth of the RC low-pass filter at the output of the multiplier are?

The voltmeter A voltmeter is available for the measurement of dc voltages. It is advisable to cross-check its output with the oscilloscope.

The high-frequency oscillator. Using a high-frequency oscillator, we determine the resonance frequency of the oscillating circuit. At the same time, we will use it to generate a reference signal (see section 4.2.3). It can generate harmonic signals with frequencies from 0.001 Hz to 1.5 MHz. The amplitude of the output signal can be controlled by either changing the amplitude knob or changing the selected attenuation.

Task: Determine experimentally the range of voltage amplitudes the oscillator can deliver at the various attenuation levels.

The LRC-bridge. This instrument can measure resistances (R), capacitances (C), and inductances (L). A manual for its operation is provided.

The oscilloscope. It is advisable to use the oscilloscope for checking the oscillating signals in the circuit. This is a good opportunity to practice the use of an oscilloscope, which is a standard experimental diagnostic tool in every research lab.

Cabling and connections. Most connections between the instruments are made with shielded coaxial cables connecting via BNC connectors. If not, there are adapters ready to use.

3 Quantitative description of the measurement principle

Description of the noise generator. The equivalent circuit shown in Fig. 2(b) is mathematically described by

$$\ddot{U}_{\mathrm{A}} + \frac{1}{\tau}\dot{U}_{\mathrm{A}} + \omega_0^2 U_{\mathrm{A}} = \frac{1}{C_{\mathrm{tot}}}\dot{I}(t), \tag{5}$$

where

$$\omega_0 = \frac{1}{\sqrt{LC_{\mathrm{tot}}}} = 2\pi f_{\mathrm{res}}, \quad \text{and} \quad \tau = RC_{\mathrm{tot}}.$$

Equation (5) describes a driven damped harmonic oscillator with resonance frequency f_{res} and damping rate $1/\tau$. The driving signal is the time-derivative of the diode current fluctuating in time (due to the shot noise). The solution of eq. (5) may equivalently be described either in the frequency, or in the time domain. Both descriptions are outlined below.

3.1 Time domain description of the circuit

The LCR-circuit. The response of a resonant circuit to an arbitrary excitation in the time domain can be found by looking at the auxiliary problem of the circuit response $G(t,t')$ to an excitation pulse at time t'. The corresponding equation is

$$\partial_t^2 G(t,t') + \frac{1}{\tau}\partial_t G(t,t') + \omega_0^2 G(t,t') = \alpha\delta(t-t'),$$

with α being the strength of the excitation pulse. We seek solutions of this equation for the initial condition $G(t,t') = 0$ for $t < t'$, i.e., we consider the circuit to be quiet before the excitation pulse arrives. The solution is then given by[3]

$$G(t,t') = \frac{\alpha}{\omega} e^{-(t-t')/2\tau} \sin\left[\omega(t-t')\right] \theta(t-t'), \tag{6}$$

where $\theta(t)$ is the Heaviside step function. This equation describes a damped oscillation with frequency

$$\omega = \sqrt{\frac{1}{LC} - \frac{1}{(2RC)^2}} = \sqrt{\omega_0^2 - \left(\frac{1}{2\tau}\right)^2}$$

[3]This solution is called Green's function of the problem. Engineers would call it the pulse response function of the circuit.

initiated by the pulse at time $t = t'$. The solution of eq. (5) is then found to be

$$U_{\mathrm{A}}(t) = \frac{1}{\omega C_{\mathrm{tot}}} \int_{-\infty}^{t} dt' e^{-(t-t')/2\tau} \sin\left[\omega(t-t')\right] \dot{I}(t').$$

Partial integration gives the result

$$U_{\mathrm{A}}(t) = \frac{1}{C_{\mathrm{tot}}} \sqrt{1 + \frac{1}{4\omega^2\tau^2}} \int_{-\infty}^{t} dt' e^{-(t-t')/2\tau} \cos\left[\omega(t-t') + \beta\right] I(t'), \quad (7)$$

where $\tan\beta = 1/\omega\tau$. This equation describes, how the resonant circuit transforms the fluctuating current into a fluctuating voltage signal. Note that for the circuit in our experiment $\omega\tau \gg 1$, i.e., damping is low.

Action of the voltage amplifier. The voltage amplifier takes the signal $U_{\mathrm{A}}(t)$ of the resonant circuit as its input and amplifies it by a factor A giving the output voltage

$$U_{\mathrm{amp}}(t) = \frac{A}{C_{\mathrm{tot}}} \sqrt{1 + \frac{1}{4\omega^2\tau^2}} \int_{-\infty}^{t} dt' e^{-(t-t')/2\tau} \cos\left[\omega(t-t') + \beta\right] I(t'). \quad (8)$$

Action of the rectifier. The rectifier takes the signal $U_{\mathrm{amp}}(t)$ as the input, multiplies it with itself, and low-pass filters the result. The output signal is either the voltage $U_z(t)$ that can be measured with the voltmeter (or the oscilloscope), or the current I_z that can be read from the display on the multiplier. Both output signals are completely equivalent. In the time domain, this means that the reading of the ammeter is

$$I_z = G_I \Gamma \int_{-\infty}^{t} dt' U_{\mathrm{amp}}^2(t') e^{-\Gamma(t-t')},$$

where Γ is the inverse time constant of the low-pass filter, and G_I is the gain factor of the rectifier. A similar relation holds for U_z, but the gain factor G_V will have different units.

Task: What are the units of G_I and G_V in the two cases where the output is a dc voltage or a dc current?

By the design of the voltmeter (or the multiplier low-pass filter) the time constant is very long (in particular, long compared to τ), such that Γ is very small. As a result, the above integral is essentially a time average of $U^2_{\rm amp}(t)$, meaning

$$I_z = G_I \langle U^2_{\rm amp}(t)\rangle. \tag{9}$$

In order to find this average, we make use of Campbell's theorem [11] (see Appendix A). To this end we assume that every electron i transferred from the cathode to the anode at $t = t_i$ generates a current pulse $I(t-t_i)$ such that the total current is given by

$$I(t) = -|e| \sum_i \delta(t - t_i), \tag{10}$$

where e is the elementary charge. We assume the electron transfers to be independent of each other in the saturation region of the tube. As a consequence, the times t_i are completely random and uncorrelated. The time averaged current $\langle I\rangle$ is, however, proportional to the average number of electrons being transferred per unit time, i.e.,

$$\langle I\rangle = -|e|n, \tag{11}$$

where n is the transfer rate of electrons.

Each individual current pulse generates a response at the output of the oscillating circuit, which has the form of a damped harmonic oscillation. We see this by inserting eq. (10) into eq. (8), giving

$$U_{\rm amp}(t) = \underbrace{\frac{-|e|A}{C_{\rm tot}}\sqrt{1+\frac{1}{4\omega^2\tau^2}}}_{:=U_0} \sum_i e^{-(t-t_i)/2\tau} \cos\left[\omega(t-t_i)+\beta\right] \theta(t-t_i).$$

The voltage $U_{\rm amp}(t)$ at the output of the amplifier is equal to the superposition of voltage pulses starting at different times $t_i < t$. An individual voltage pulse starting at time zero has the form of the damped harmonic oscillation

$$U(t) = U_0 e^{-t/2\tau} \cos\left[\omega t + \beta\right] \quad \text{for } t \geq 0. \tag{12}$$

In such a situation, Campbell's theorem [11] (Appendix A) gives the average $\langle U^2_{\rm amp}(t)\rangle$ as

$$\langle U^2_{\rm amp}(t)\rangle = n \cdot \int_0^\infty U^2(t)dt. \tag{13}$$

Inserting eqs. (12) into (13) and making use of $\omega\tau \gg 1$ results after some algebra in

$$\langle U^2_{\rm amp}(t)\rangle = \frac{A^2e^2n\tau}{2C^2_{\rm tot}} = A^2\frac{|e|\,|\langle I\rangle|\,R}{2C_{\rm tot}}.$$

Inserting this equation into eq. (9) gives the final result

$$\boxed{I_z = G_I A^2 \times R^2 \times 2|e|\,|\langle I\rangle| \times \frac{1}{4\tau} = \frac{G_I A^2 R|e|\,|\langle I\rangle|}{2C_{\text{tot}}}.} \tag{14}$$

3.2 Frequency domain description of the circuit

The same result can be obtained, by seeking the solution of eq. (5) in the frequency domain. Doing this leads to further interesting insights into the problem. Fourier transforming eq. (5) leads to

$$\hat{U}_{\text{A}}(\omega) = \frac{i\omega/C_{\text{tot}}}{\omega_0^2 - \omega^2 + i\omega/\tau}\hat{I}(\omega). \tag{15}$$

This equation describes the resonant response of the circuit to a driving current. It is the analogue of eq. (7). Note that ω is *not* the resonance frequency of the circuit here, but can take any values. For example, for $\omega = \omega_0$ we find $\hat{U}_{\text{A}}(\omega_0) = R\hat{I}(\omega_0)$, i.e., the response is purely resistive.

Action of the voltage amplifier. Assuming that the voltage amplifier has a flat frequency response in the frequency range around the resonance of the resonant circuit, the output signal after amplification by a factor A is

$$\hat{U}_{\text{amp}}(\omega) = \frac{iA\omega/C_{\text{tot}}}{\omega_0^2 - \omega^2 + i\omega/\tau}\hat{I}(\omega). \tag{16}$$

This is the frequency-domain analogue to eq. (8).

Action of the rectifier. Multiplication of the signal in the time domain is equivalent to a convolution integral in the frequency domain leading to the voltage $U_z(\omega)$ at the output of the multiplier given by

$$\begin{aligned} U_z(\omega) &= \frac{G_V}{2\pi}\int_{-\infty}^{+\infty} d\omega' \hat{U}_{\text{amp}}(\omega')\hat{U}_{\text{amp}}(\omega-\omega') \\ &= -\frac{G_V A^2}{2\pi C_{\text{tot}}^2}\int_{-\infty}^{+\infty} d\omega' \frac{\omega'\hat{I}(\omega')}{\omega_0^2 - \omega'^2 + i\omega'/\tau}\frac{(\omega-\omega')\hat{I}(\omega-\omega')}{\omega_0^2 - (\omega-\omega')^2 + i(\omega-\omega')/\tau}, \end{aligned} \tag{17}$$

where G_V is the multiplier voltage gain factor. The low-pass filter at the output of the multiplier averages the squared voltage and converts it into a current. It therefore essentially selects the zero frequency component $U_z(0)$ giving

$$I_z = \frac{G_I A^2}{2\pi C_{\mathrm{tot}}^2} \int_{-\infty}^{+\infty} d\omega \frac{\omega^2 \hat{I}(\omega)\hat{I}(-\omega)}{(\omega_0^2 - \omega^2)^2 + \omega^2/\tau^2}. \tag{18}$$

Since $I(t)$ is a real valued signal, $\hat{I}(-\omega) = \hat{I}^\star(\omega)$. The product

$$\tilde{S}_I(\omega) = \hat{I}(\omega)\hat{I}^\star(\omega) = |\hat{I}(\omega)|^2$$

is the (bilateral) power spectral density of the current shot noise (Wiener–Chinchin theorem). At the same time, the integrand is even in ω. This leads to

$$I_z = \frac{G_I A^2}{2\pi C_{\mathrm{tot}}^2} \int_{0}^{+\infty} d\omega \frac{\omega^2 2\tilde{S}_I(\omega)}{(\omega_0^2 - \omega^2)^2 + \omega^2/\tau^2}. \tag{19}$$

Note that we now integrate over positive frequencies only. The single sided spectral density

$$S_I(\omega) = 2\tilde{S}_I(\omega)$$

is called the unilateral power spectral density of the current noise. We can now write

$$I_z = \frac{G_I A^2}{2\pi C_{\mathrm{tot}}^2} \int_{0}^{+\infty} d\omega \frac{\omega^2 S_I(\omega)}{(\omega_0^2 - \omega^2)^2 + \omega^2/\tau^2}. \tag{20}$$

We see that the resonant circuit acts as to select from the power spectral density $S_I(\omega)$ a narrow frequency band around the resonance frequency ω_0 as shown in Figure 4. The output signal $U_z(0)$ is proportional to the noise power

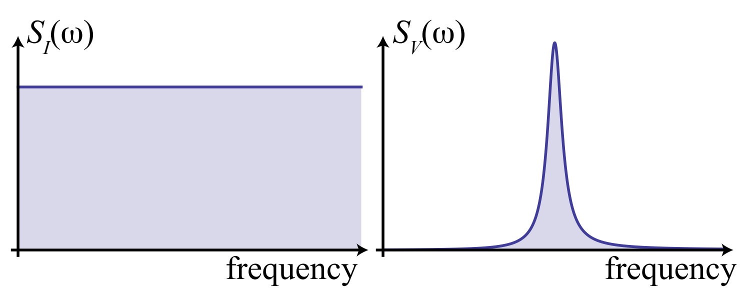


Figure 4: Left: White noise power spectral density of the shot-noise current. Right: Voltage noise power at the output (A) of the resonant circuit. The shaded area below this curve corresponds to the measured mean squared voltage noise.

spectral density $S_I(\omega)$ integrated over this frequency band (see shaded area in Fig. 4, right plot). Because the power spectral density for shot noise $S_I(\omega)$ is independent of frequency in the range of interest around the resonance of the resonant circuit, we talk about white noise. In our case $S_I(\omega) = S_0 = 2|e||\langle I\rangle|$ [c.f., eq. (4)]. Inserting this result into eq. (18) leads to

$$I_z = \frac{G_I A^2 R^2}{2\pi\tau} S_0 \int_0^{+\infty} d(\omega\tau) \frac{(\omega\tau)^2}{\left((\omega_0\tau)^2 - (\omega\tau)^2\right)^2 + (\omega\tau)^2}. \tag{21}$$

The frequency integral evaluates to $\pi/2$ giving the result

$$\boxed{I_z = G_I A^2 R^2 S_0 \frac{1}{4\tau} = G_I A^2 \times R^2 \times 2|e|\,|\langle I\rangle| \times \Delta f,} \tag{22}$$

in complete correspondence to the time-domain result (14). The quantity $\Delta f = 1/4\tau = 1/4RC_{\text{tot}}$ may be interpreted as the effective bandwidth of the resonant circuit. The product $\langle \Delta I^2\rangle = 2|e||\langle I\rangle|\Delta f$ is therefore the mean current fluctuation within the resonant circuit bandwidth. The factor R^2 converts this current fluctuation into a mean voltage fluctuation $\langle U_{\text{A}}^2\rangle = R^2 \times 2|e||\langle I\rangle|\Delta f$ at the amplifier input. The amplification factor $G_I A^2$ amplifies this signal to give the output current I_z.

Equation (22) allows the determination of the elementary charge $|e|$ from measurable quantities $\langle I\rangle$, I_z, $G_I A^2$, R, and C_{tot}.

4 Experimental procedure

4.1 Goal of the measurements

The elementary charge is to be determined by measuring the shot noise for at least ten different diode currents I between 5 and 25 mA. Using formula (22), the elementary charge can be determined by measuring the other unknown quantities (see section 4.2). Each quantity that is directly measurable should be measured repeatedly (at least a few times) in order to estimate the true value and its uncertainty. Work out carefully how these uncertainties lead to the final uncertainty of the result for the elementary charge.

In order to avoid unnecessary systematic errors, short cables should be used in all measurements. Between the amplifier and the rectifier, equally long cables have to be used. BNC connectors should not be left open, since this could influence the measurement.

4.2 Measurement of the individual quantities

General advice. After turning on the various units of the equipment they should be given time to equilibrate thermally before commencing with any measurement. The current of the noise generator is a good indicator for this process.

It is generally advisable to keep the signal amplitudes well within the range of the measuring units. The HF oscillator output and the amplifier gain should be adjusted such that the measurements will neither be made in the lowest range of the voltmeter, nor in the saturation regime of the rectifier display. The oscilloscope is an ideal tool for checking the quality of the signals. Coarse and qualitative test measurements can tell you, whether the chosen setup and the corresponding parameters yield consistent individual results and will therefore lead to an acceptable value of the elementary charge. Only after such tests have been made, precise and systematic measurements should be started.

Task: While reading through the following measurement procedures, develop a good measurement strategy. Which of the required measurements of ρ, C_{tot}, R, and $\langle V_{\mathrm{amp}}^2 \rangle$ are independent, which measurements can or should be combined for most precise results? In which sequence will you perform these measurements? Discuss your strategy with the supervisor.

4.2.1 Measurement of C_{tot}

The capacitance C_{tot} corresponds to the oscillating circuit's capacitance (capacitor and diode capacitance), the capacitance of the amplifier input and the capacitances of the connection cables (coaxial cables) from the noise generator to the amplifier. Therefore, the amplifier has to be connected when C_{tot} is measured.

Excite the oscillating circuit with the oscillator connected in C and the amplifier connected in A. The resonance frequency ω_1 of the resonant circuit can now be determined.

Task:

1. Why can we not excite the resonator by connecting the oscillator directly in A?

2. Does it make a difference whether we use the $50\,\Omega$ or the $600\,\Omega$ output of the oscillator?

In the next step a known capacitance C_1 is connected parallel to the oscillating circuit, i.e., between A and ground (B) (see Fig. 5). The resonance frequency ω_2 can be determined.

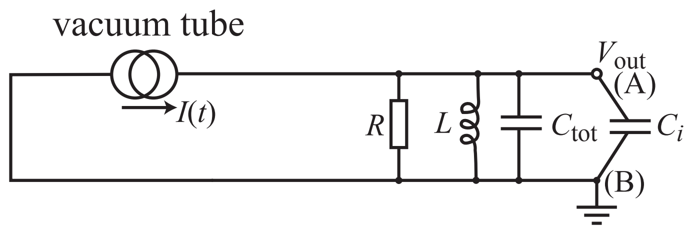


Figure 5: Measurement of the effective capacitance C_{tot}

Since the two resonance frequencies are given by $\omega_1 = 1/\sqrt{LC_{\text{tot}}}$ and $\omega_2 = 1/\sqrt{L(C_{\text{tot}} + C_1)}$, the effective capacitance C_{tot} (and the inductance L) can be calculated from the knowledge of ω_1, ω_2 and C_1.

The capacitance C_{tot} does not vary for different diode currents I. It is therefore sufficient to measure C_{tot} for one current only (but it still has to be measured repeatedly).

Task: Check this statement experimentally on a qualitative level.

4.2.2 Measurement of R

Measurement of ρ. Before determining R, it is necessary to measure the resistance ρ (see Fig. 6). To this end, the RCL-bridge hast to be connected to terminal (C). Terminal (C) is the inner conductor of a BNC connector. The outer conductor of this connector is ground (B). For operating the LCR-bridge, follow the instructions in the provided manual.

Measurement of R. The damping resistance R of the oscillating circuit changes with the diode current. The measurement of R works as follows: connect the oscillator to terminal (C), and the amplifier to terminal (A). The excitation frequency has to be set to the resonance frequency. Calibrate the first amplifier stage and extract the gain A_1.

The oscillating circuit is now excited with an amplitude U_1 that you choose to be not too low and not too high. Measure this amplitude with the voltmeter

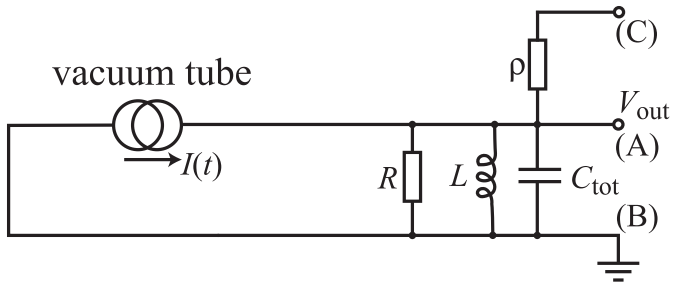


Figure 6: Determination of R.

before or after connecting the oscillator to terminal (C). With the oscillator connected to (C), measure the output of the first amplifier stage with the voltmeter and determine the voltage U_2 at the amplifier input by dividing the value by A_1.

Since the current I through both resistances is the same, it follows that

$$R = \frac{U_2}{U_1 - U_2} \cdot \rho \tag{23}$$

Task:

1. Why is it important to set the excitation amplitude not too low? What determines the upper limit of the excitation amplitude?
2. Why is it important to do this measurement precisely at the resonance frequency?
3. Why do we need to use the amplifier for this measurement? Would it not be much easier to measure U_1 with the voltmeter directly at (C) and U_2 at (A)?

4.2.3 Measurement of $\langle U_A^2(t) \rangle$

The indicator of the rectifier shows a quantity which is proportional to the shot noise, but there is an unknown calibration factor $G_I A^2$ (see eq. (22)). As a consequence, $\langle U_A^2(t) \rangle$ cannot be read directly from the display. In order to calibrate $G_I A^2$ properly, you have to use a reference signal of well-known amplitude (generated by the high frequency oscillator) and make a note of the corresponding reading of the indicator of the rectifier. To do it correctly, first find a range of amplified voltage for which the indicator on the rectifier does not saturate. For this, consider making a quick test using the highest diode current I and setting the amplification factor such that the amplitude I_z on the ammeter of the rectifier is reasonably high. Once the amplification factor A is set, it should not be changed anymore for the remaining experiments to achieve most precise results.

Calibration of the amplifier and multiplier. The reference signal you can use for calibration is the sinusoidal signal provided by the HF oscillator. Feed this signal at the resonance frequency into the amplifier and vary the amplitude U_0 of the signal (U_0 may either be measured with the oscilloscope or the ac-voltmeter). The range of possible amplitudes will be very low. Therefore attenuate the oscillator output signal. If needed, an additional 20 dB attenuator is found at the experiment. Calibrate the amplifier-multiplier assembly by taking a reasonable number of data points (U_0, I_z). Estimate the quantity $G_I A^2$ for your data points and make sure you obtain reasonably similar values for all points. When you do a proper data analysis later, curve fitting may help you to obtain a more precise calibration.

5 Derivation of the Schottky formula [1]

In a vacuum tube thermally excited electrons are emitted from the hot cathode, and sucked away by a large electric field. The tunneling barrier is characterized by a transmission $\mathcal{T}(E)$ depending on the energy of the impinging electron [see Fig. 7]. Well below the top of the barrier the transmission is exponentially suppressed, whereas far above the barrier it is essentially one. Close to the top of the barrier the transmission exhibits a sharp step from values $\mathcal{T}(E) \ll 1$ to $\mathcal{T}(E) \approx 1$. The work function W of the cathode material is of the order of 5 eV, the cathode temperature is of the order of 2000°C corresponding to a thermal energy $k_B T \approx 190$ meV. The occupation $f_K(E)$ of states in the cathode is given by the Fermi–Dirac distribution function. Since $k_B T \ll W$ the occupation probability close to the barrier top, where the transmission becomes appreciable, is very small and well described by the Boltzmann distribution. The states on the vacuum side of the barrier can be considered to be unoccupied because any tunneling electron is immediately sucked away. Thermionic emission of electrons from the cathode is determined by the interplay between the sharp step of the transmission function

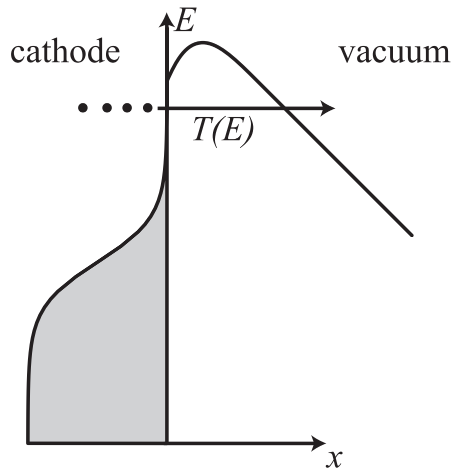


Figure 7: Schematic diagram of thermally activated tunneling from the heated cathode of the diode into the vacuum.

at the barrier top, and the exponential tail of the Boltzmann distribution. As a result, the product $f_{\mathrm{K}}(E)\mathcal{T}(E)$ shows a marked maximum near the top of the barrier, but even there the value $f_{\mathrm{K}}(W)\mathcal{T}(W) \ll 1$ (we choose the cathode Fermi energy as the energy zero). This is the characteristic situation for thermionic emission of electrons. The number of cathode states at this energy is proportional to the density of states $\mathcal{D}(W)$.

Electron transmission as a probabilistic experiment. We now describe the electron emission process from the cathode as a probabilistic experiment. Assume that within an observation time t_0, $N \propto \mathcal{D}(W)$ attempts were possible for electrons to hit the barrier. The statistics of whether such a potential attempt leads to a tunneling electron or not depends on the probability $p = f_{\mathrm{K}}(W)\mathcal{T}(W)$. The situation for an individual potential attempt is the same as in a probabilistic experiment with two possible outcomes, such as tossing a coin. Here, the two out- comes are, (1) an attempt is successful (probability p) and an electron is transmitted, or (2) an attempt is unsuccessful (probability $1-p$) and no electron is transmitted. The probability that out of the N attempts, n electrons are transmitted, is then given by the *binomial distribution*

$$P(n) = \begin{pmatrix} N \\ n \end{pmatrix} p^n (1-p)^{N-n} = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n}. \qquad (24)$$

In the case of thermionic emission from a metal, where $p \ll 1$, the binomial distribution can be well approximated by the *Poisson distribution*

$$P(n) = \frac{\mu^n}{n!} e^{-\mu}, \tag{25}$$

with the mean value $\mu \equiv \langle n \rangle = Np$ and the variance $\sigma^2 = \mu = Np$.

Average current and classical shot noise. With the Poisson distribution function, the mean electrical current is calculated to be

$$I = -\frac{|e|\langle n \rangle}{t_0} = -\frac{|e|N}{t_0} p. \tag{26}$$

It could be measured in a way in which the transmitted electrons are repeatedly counted over a time span t_0 on the anode side. The average of the number of counts is then determined from the measured counting statistics which we have assumed to be poissonian. The time span t_0 plays the role of an integration time, or equivalently, $\Delta\nu = 1/2t_0$ is the bandwidth of the measurement apparatus.

The shot noise is revealed if we consider the temporal fluctuations in the number of transmitted electrons which is related to the width of the Poisson distribution function. The average of these fluctuations is given by

$$(\delta n)^2 = \langle (n - \langle n \rangle)^2 \rangle = \mu = np. \tag{27}$$

Correspondingly, the mean fluctuations of the electrical current are

$$\left\langle \Delta I^2 \right\rangle_{t_0} = \langle I^2 \rangle - \langle I \rangle^2 = \frac{e^2 (\Delta n)^2}{t_0^2} = \frac{e^2}{t_0^2} Np = \frac{|e|}{t_0} |\langle I \rangle| = 2|e||\langle I \rangle|\Delta\nu. \tag{28}$$

The spectral density of the shot noise is

$$S_0 = 2|e||\langle I \rangle|. \tag{29}$$

6 Tasks

1. Determine the elementary charge $|e|$ from the shot noise for at least five different diode currents.

2. Determine the uncertainties of the measured quantities including the elementary charge with a clear error analysis to check the quality of the measurement. Compare your estimate for $|e|$ and its uncertainty with the literature value and judge the accuracy of your measurement.

3. (a) Derive equation (5).
   (b) How is the quality factor Q of the resonant circuit related to R, L, and C_{tot}? How big is the quality factor of the resonant circuit used in this experiment? Use secondary literature, if you need a definition of the quality factor.
   (c) Derive equation (14) using eq. (12) and Campbell's theorem in eq. (13).
   (d) In the quantitative description of the measurement circuit in the frequency domain, the action of the low-pass filter at the output of the rectifier is taken into account approximately by using the zero-frequency limit. What is the frequency domain description of a low-pass filter? Can you justify the zero-frequency limit more rigorously, using the proper low-pass filter characteristics?
   (e) Derive eq. (23).

4. Answer the following questions:
   (a) Apart from shot noise, how else can the elementary charge be determined (directly or indirectly)? What are the advantages and disadvantages of the different methods, and what is their precision?
   (b) Why does one have to set the correct current on the diode to measure R?
   (c) Why does the experiment use an oscillating circuit as the anode impedance?

A Campbell's theorem [11]

Let $F(t)$ be a function of time which is generated by superimposing identical functions $f(t-t_i)$ occurring at an average rate n_0, but at completely random and statistically independent times t_i:

$$F(t) = \sum_i f(t-t_i). \tag{30}$$

The average response at a later time t'' due to impulses in an interval dt' around time t' is then given by $f(t''-t')n_0dt'$. The constant average response of the system at time t'' due to impulses arriving during all previous time intervals is then given by

$$\langle F(t)\rangle = \int_{-\infty}^{t''} n_0 dt' f(t''-t')$$

Changing variables to $t = t'' - t'$ gives the first part of Campbell's theorem

$$\boxed{\langle F(t) \rangle = n_0 \cdot \int_0^{\infty} f(t)dt.} \quad \textbf{Campbell's theorem (a)} \tag{31}$$

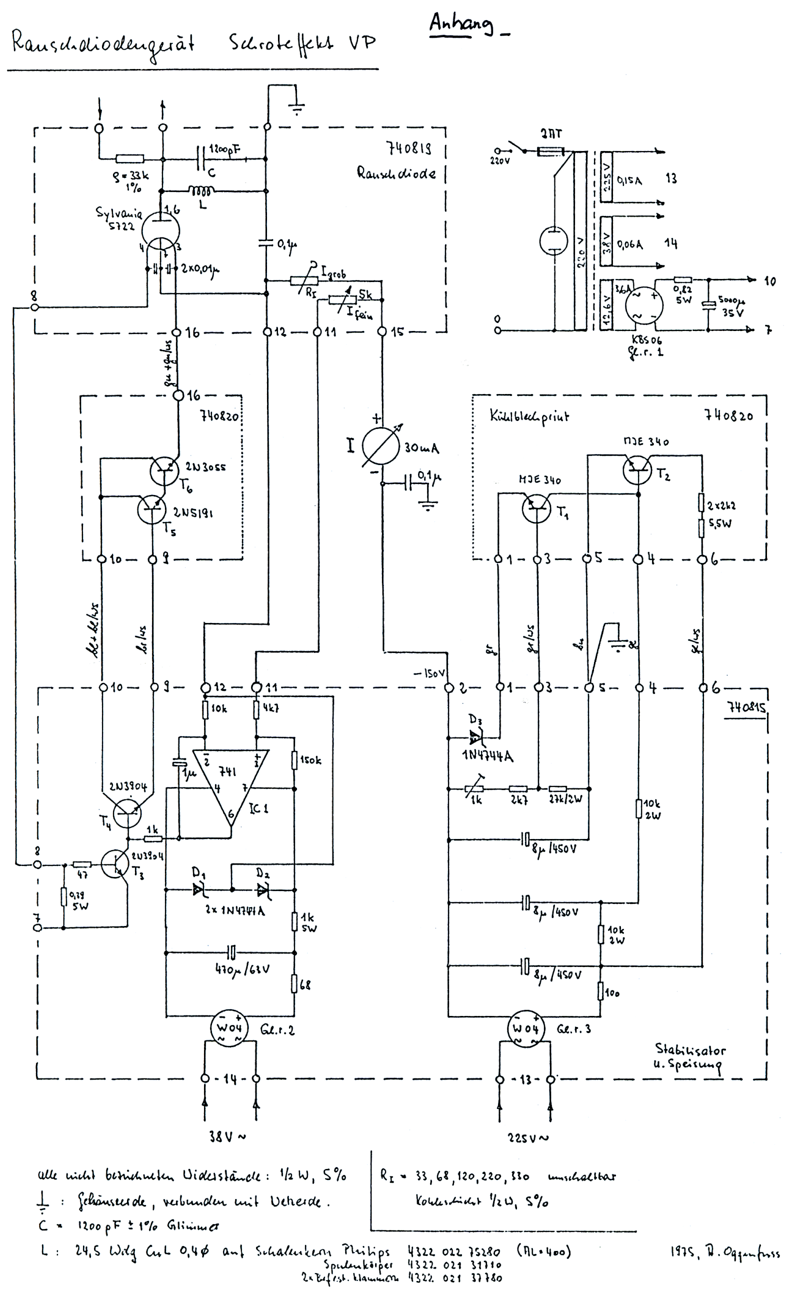
For the mean square of $F(t)$, we have correspondingly

$$\langle F^2(t) \rangle = \int_{-\infty}^{t''} n_0 dt' f^2(t'' - t')$$

giving, with the same substitution used above, the second part of Campbell's theorem

$$\boxed{\langle F^2(t) \rangle = n_0 \int_0^{\infty} dt f^2(t).} \quad \textbf{Campbell's theorem (b)}$$

B Circuit diagram of the noise module



Anode current stabilization

Due to the high ampflification of the operational amplifier 741, $u^- \approx u^+$. Therefore, a current $i = 15\text{V}/150\text{k}\Omega = 0.01$ mA flows through $R_1 = 150$ kΩ. The current $i^+ \ll 0.1$mA, meaning that i has to flow through R_2 as well. Thus, a voltage $u_{12} = iR_2$ arises between 1 and 2 and the anode current is stabilized to $I = u_{12}/R_I = iR_2/R_I$. If I would be too low, then $u_{12} < iR_2$ and therefore $u^+ > u^-$, but then u_{out} grows and would turn-on the Darlington combination stronger which would increase the tube heating power and therefore the anode current would rise too.

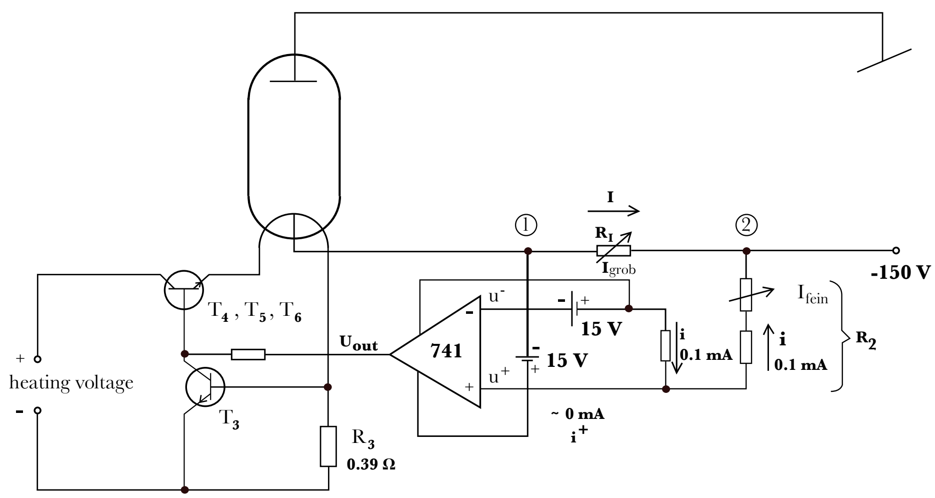
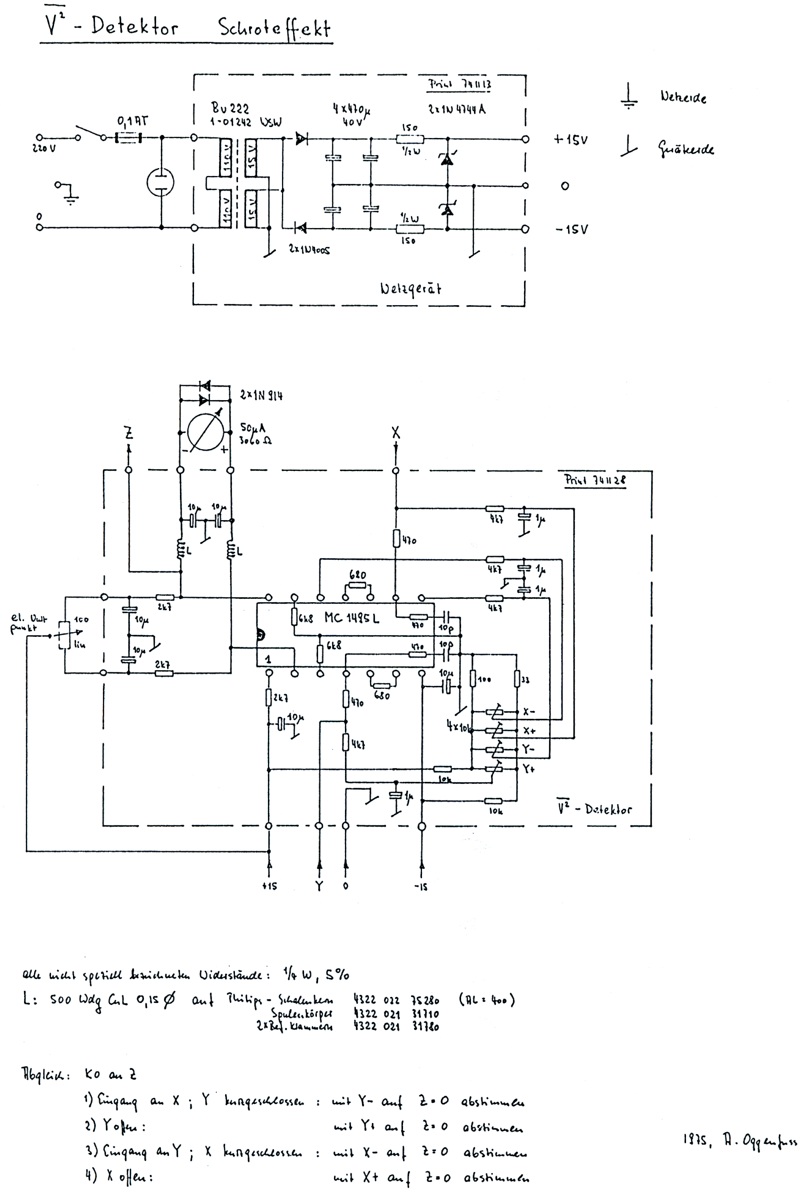
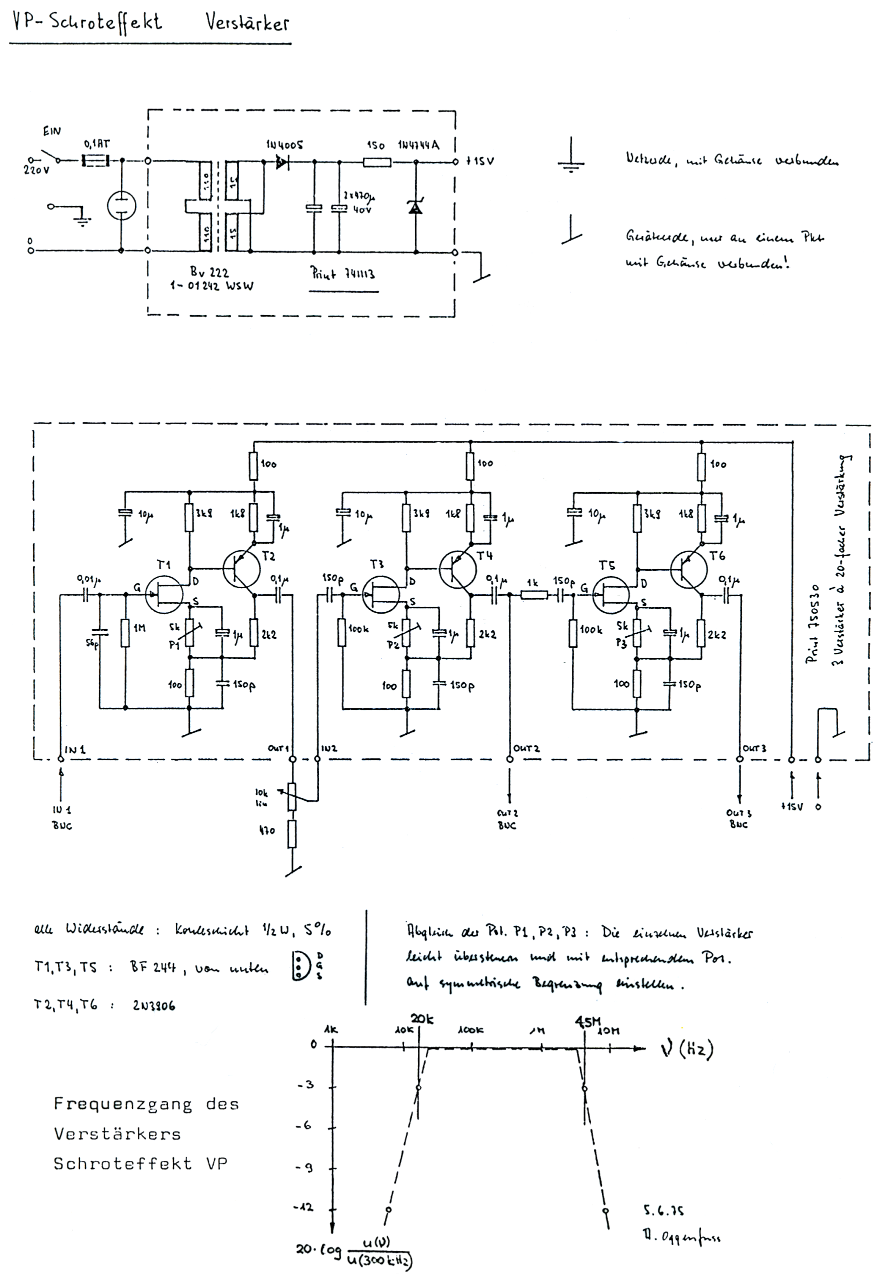


Figure 8: Scheme of the anode current stabilization circuit.

C Circuit diagram of the rectifier module



D Circuit diagram of the voltage amplifier



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