

Nr.:

Polarization of light

Laboratory work

Version 1.6.2

Yohan Barbarin, Rev. by Stephan Smolka

Department of Physics

ETH Zurich

Januar 2014

Tasks of the laboratory work

In this laboratory work, students will study the polarization of light in different physical experiments. In particular, Malus' Law and Brewster's Law will be studied. Measurements of reflectivity and transmittance of a glass plate will be compared with results predicted by the Fresnel Equations. The students will use a pile of glass plates to build a polarizer. In the second part of the experiment, a polarimeter will be realized to measure the concentration of glucose in a solution (saccharimetry). Finally, the nonlinear electro-optical effect generated by a Pockels Cell will be exploited to modulate the intensity of the light.

Table of Contents

Tasks of the laboratory work	1
1. Theoretical Background	3
1.1. Types of Polarization	3
1.2. Polarized Light and Malus' Law	5
1.3. The Fresnel Equations	6
1.4. Transmittance and Reflectivity of a Glass Plate	10
1.5. Isotrop and Anisotrop Media/ Non-Linear Effects	13
2. Polarizers	14
2.1. Basic and complete relations of polarizer	14
2.2. Polarization by Nonnormal-Incidence Reflection (Pile of Plates)	16
3. Applications	20
3.1. Polarization of light as analyzer of material	20
3.2. Liquid crystal displays (LCDs)	21
3.3. Pockels cells and Electro-Optic Modulators (EOM)	22
4. Experimental Studies	24
4.1. Demonstration of the Malus' Law	24
4.2. S and P transmission and reflection through and on a glass plate	24
4.3. A pile of glass plates polarizer	25
4.4. Construction of an EOM using a Pockels cell.	26
4.5. Measurement of sugar concentration (not available)	27
5. Annexes	28
5.1. Polaroid Material	28
5.2. Glan–Foucault prism	28
5.3. Nicol prism	29
5.4. Optical and electrical risks	30
5.5. Procedure to align a Pockels cell	30
6. References	32

1. Theoretical Background

1.1. Types of Polarization

The form of polarization of light can be quite complex. However, for most design situations there are a limited number of types that are needed to describe the polarization of light in an optical system. Figure 1 shows the path traced by the electric field during one full cycle of oscillation of the wave ($T=1/\nu$), where ν is the frequency of the light, for a number of different types of polarization. Figure 1(a) shows linear polarization, where orientation of the electric field vector of the wave does not change with time as the field amplitude oscillates from a maximum value in one direction to a maximum value in the opposite direction. The orientation of the electric field is referenced to some axis perpendicular to the direction of propagation. In some cases, it may be a direction in the laboratory or optical system, and it is specified as horizontally or vertically polarized or polarized at some angle to a coordinate axis. Because the electric field is a vector quantity, electric fields add as vectors. For example, two fields, E_x and E_y linearly polarized at right angles to each other and oscillating in phase (maxima for both waves occur at the same time), will combine to give another linearly polarized wave, shown in Figure 1(b), whose direction ($\tan \theta = E_y / E_x$) and amplitude ($\sqrt{E_x^2 + E_y^2}$) are found by addition of the two components. If these fields have 90° phase difference (the maximum in one field occurs when the other field is zero), the electric field of the combined fields traces out a circle. The result is called circularly polarized light. If the phase difference is not 90° , the projection is an ellipse during one cycle, as shown in Figure 1(c). The result is called elliptically polarized light. The eccentricity of the ellipse is the ratio of the amplitudes of the two components. Since the direction of rotation of the vector depends on the relative phases of the two components, this type of polarization has a handedness to be specified. If the electric field coming from a source toward the observer rotates counterclockwise, the polarization is said to be left-handed. Right-handed polarization has the opposite sense, clockwise. This nomenclature applies to elliptical as well as circular polarization. Light whose direction of polarization does not follow a simple pattern such as the ones described here is sometimes referred to as unpolarized light.

This can be somewhat misleading because the field has an instantaneous direction of polarization at all times, but it may not be easy to discover what the pattern is. A more descriptive term is randomly polarized light.

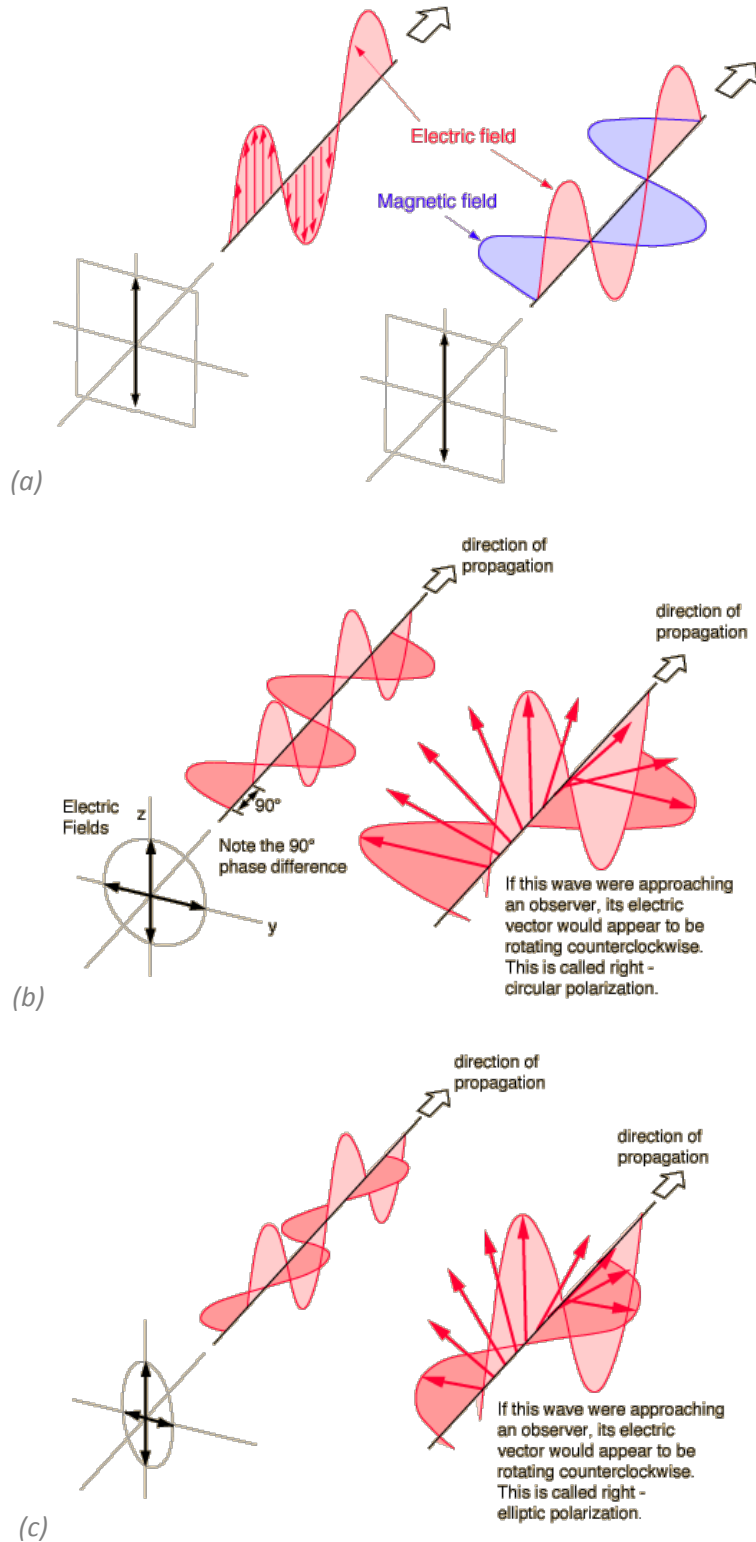


Figure 1: Three special polarization orientations: (a) linear, (b) circular and (c) elliptical

1.2. Polarized Light and Malus' Law

In an electromagnetic wave the magnetic field H is orthogonal to the electric field E (see figure 1 (a)). In an isotropic, non-magnetic medium both fields are perpendicular to the so-called wave vector k . By convention, the direction of the electric field defines the polarization of a wave. If the direction of the electric field changes randomly over time, the wave is called unpolarized. If the electric field vector is confined to a single plane, the light is linearly polarized.

A polarizer is a device that converts unpolarized or mixed-polarized light into (usually linearly) polarized light. Many polarizers achieve this by exploiting the birefringent properties of certain crystals. If a polarizer is applied to an already polarized light, it is called an analyzer.

Linearly polarized light is never perfectly polarized. That's why the contrast C is defined, which specifies the ratio between the intensity of the linearly polarized light and the intensity of the thereto perpendicular polarization. A helium-neon-laser typically has a contrast of 1:500. Commercial 3M polarizer films produce linearly polarized light with a contrast of 1:500 when Glan-Thompson type s of polarizers produce linearly polarized light with a contrast of 1:100,000. (See Annexes 5.1, 5.2 and 5.3)

When linearly polarized light is analyzed (rotation of a polarizer around the direction of propagation, as shown in figure 2), the amplitude of the entering light is projected onto the axis of the polarizer. If Ψ is the angle between the light's initial polarization and the axis of the polarizer, the emerging light has a linear polarization, which is parallel to the axis of the polarizer and has an amplitude of $A_{out} = |A_{in} \cos \Psi|$. The intensity received by a detector is therefore given by:

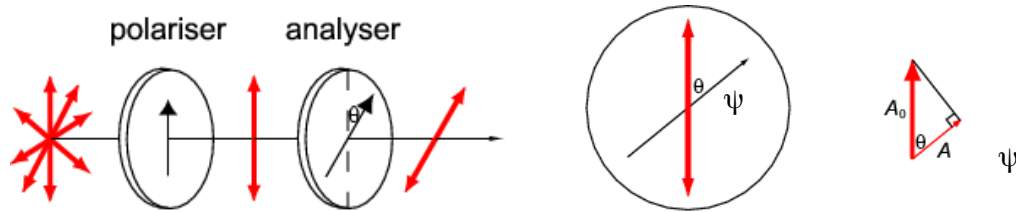


Figure 2: Polarised beam incident on a second polarizer & vertically polarised beam incident on an analyser

$$I(\Psi) = A_{out}^2(\Psi) = I_{in} \cdot \cos^2 \Psi = \frac{1}{2} I_{in} \cdot (1 + \cos 2\Psi) \quad (1)$$

This law for the analysis of a linearly polarized light is called the Malus' Law.

1.3. The Fresnel Equations

The Fresnel equations, which can be derived from Maxwell's equations, describe the behavior of linearly polarized light at a phase boundary, which constitutes a change of refractive index between media. The so-called plane of incidence is given by the incoming light wave and the perpendicular to the surface of the medium (Figure 4). One has to differentiate between polarization parallel and polarization orthogonal to the plane of incidence. The component of the electric field parallel to the plane is termed p-like (parallel) and the component perpendicular to this plane is termed s-like (from *senkrecht*, German for perpendicular). Light with a p-like electric field is said to be p-polarized, tangential plane polarized, or is said to be a transverse-magnetic (TM) wave. Light with an s-like electric field is s-polarized, also known as sagittal plane polarized, or it can be called a transverse-electric (TE) wave.

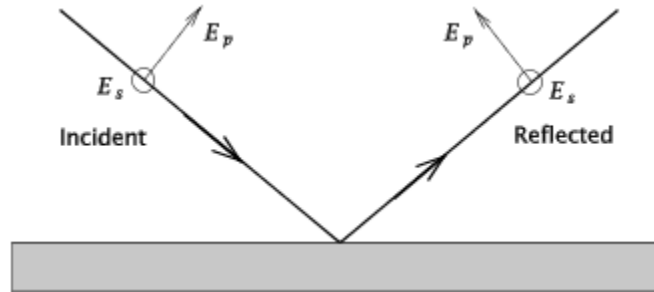


Figure 4: Reflection of a plane wave from a surface perpendicular to the page. The p-components of the waves are in the plane of the page, while the s components are perpendicular to it.

The basic idea for a derivation of the Fresnel equations is that the conservation of energy applies to a reflection and therefore the sum of the radiant power at a phase boundary is constant. The radiant power Φ of a light beam is equal to the product of energy density S and the cross section of the beam. The energy density in turn is proportional to the square of the strength of the electric field of the wave:

$$S = \sqrt{\frac{\epsilon}{\mu_0}} \cdot E^2 = \frac{n}{\sqrt{\mu_0}} \cdot I \quad (2)$$

Where μ_0 is the vacuum permeability and $n = \sqrt{\mu\epsilon} = \sqrt{\mu}$ is the refraction index for a non-magnetic medium. For the phase boundary shown in Figure 5, α being the angle of incidence and β being the diffraction angle, the equation $\Phi_e = \Phi_r + \Phi_t$ implies that:

$$S_e \cdot A \cos \alpha = S_r \cdot A \cos \alpha + S_t \cdot A \cos \beta \quad (3)$$

$$I_e \cdot n_1 A \cos \alpha = I_r \cdot n_1 A \cos \alpha + I_t \cdot n_2 A \cos \beta \quad (4)$$

The angle of diffraction is determined by Snel law:

$$n_2 \cdot \sin \square = n_1 \sin \square \quad (5)$$

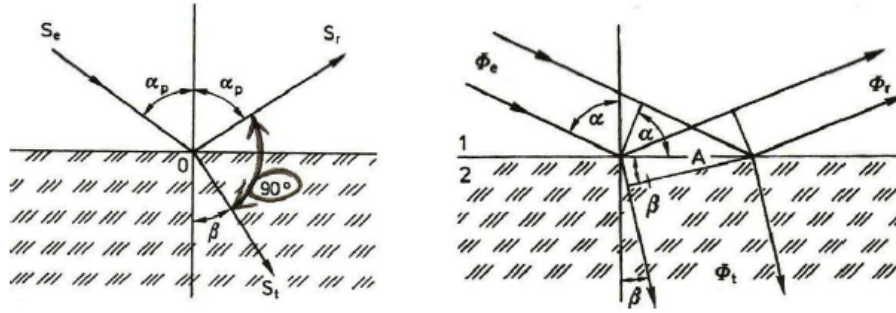


Figure 5: left: Reflection and transmittance of a light beam in the special case of the Brewster angle. Right Reflection and transmittance of a wide beam for the derivation of the Fresnel equation.

History of the law:

Ptolemy, of the Thebaid, had found a relationship regarding refraction angles, but which was inaccurate for angles that were not small. Ptolemy was confident he had found an accurate empirical law, partially as a result of fudging his data to fit theory

Snell's law was first accurately described in a mathematical form by Ibn Sahl, of Baghdad, in the manuscript *On Burning Mirrors and Lenses* (984). He made use of it to work out the shapes of lenses that focus light with no geometric aberrations, known as anastigmatic lenses.

It was rediscovered by Thomas Harriot in 1602, who however did not publish his results although he had corresponded with Kepler on this subject. In 1621, Willebrord Snellius (Snel) derived a mathematically equivalent form, that remained unpublished during his lifetime. René Descartes independently derived the law using heuristic momentum conservation arguments in terms of sines in his 1637 treatise *Discourse on Method*, and used it to solve a range of optical problems. Rejecting Descartes' solution, Pierre de Fermat arrived at the same solution based solely on his principle of least time.

According to Dijksterhuis, "In *De natura lucis et proprietate* (1662) Isaac Vossius said that Descartes had seen Snell's paper and concocted his own proof. We now know this charge to be undeserved but it has been adopted many times since." Both Fermat and Huygens repeated this accusation that Descartes had copied Snell. In French, Snell's Law is called "la loi de Descartes" or "loi de Snell-Descartes."

Using the relation $n_{21} = n_2 / n_1$ for the relative refraction index, which is changing according to 1→2, and the equation

$$f_{21} = \frac{A \cdot \cos \beta}{A \cdot \cos \alpha} = \frac{\cos \beta}{\cos \alpha} = \frac{\sqrt{n_{21}^2 - \sin^2 \alpha}}{n_{21} \cos \alpha} \quad (6)$$

which defines the quotient of the cross sections of medium 2 and medium 1, it follows from (4) that:

$$I_e = I_r + n_{21} f_{21} \cdot I_t = R_{21} I_e + T_{21} I_e \quad (7)$$

Here, the reflectivity R and transmittance T , both referring to the intensity, were introduced. Using the same nomenclature, it follows that if the refractive index is now changing according to $2 \rightarrow 1$, that:

$$S_e \cdot A \cos \beta = S_r \cdot A \cos \beta + S_t \cdot A \cos \alpha \quad (8)$$

$$I_e \cdot n_2 A \cos \beta = I_r \cdot n_2 A \cos \beta + I_t \cdot n_1 A \cos \alpha \quad (9)$$

Which can be written as:

$$I_e = I_r + n_{12} f_{12} \cdot I_t = R_{12} I_e + T_{12} I_e \quad (10)$$

Further observations describe how the amplitudes of the electric fields - either polarized in s-direction (orthogonal/'senkrecht' to the plane of incidence) or in p-direction (parallel to the plane of incidence) - change at the phase boundary. In summary, the intensities of the two directions of polarization, when the refractive index is changing according to $1 \rightarrow 2$ or $2 \rightarrow 1$, can be described by the equations displayed in Tables 1, 2 and 3.

Intensity for $n : 1 \rightarrow 2$	Reflectivity R_{21}	Transmittance T_{21}
s-polarization	$\left(\frac{1 - n_{21} f_{21}}{1 + n_{21} f_{21}} \right)^2$	$n_{21} f_{21} \cdot \left(\frac{2}{1 + n_{21} f_{21}} \right)^2$
p-polarization	$\left(\frac{n_{21} - f_{21}}{n_{21} + f_{21}} \right)^2$	$n_{21} f_{21} \cdot \left(\frac{2}{n_{21} + f_{21}} \right)^2$

Table 1: Reflectivity and Transmittance through an interface between 2 media for the light going from the medium 1 (refractive index n_1) to the medium 2 (refractive index n_2)

Intensity for $n : 2 \rightarrow 1$	Reflectivity R_{12}	Transmittance T_{12}
s-polarization	$\left(\frac{1 - n_{12} f_{12}}{1 + n_{12} f_{12}} \right)^2$	$n_{12} f_{12} \cdot \left(\frac{2}{1 + n_{12} f_{12}} \right)^2$
p-polarization	$\left(\frac{n_{12} - f_{12}}{n_{12} + f_{12}} \right)^2$	$n_{12} f_{12} \cdot \left(\frac{2}{n_{12} + f_{12}} \right)^2$

Table 2: Reflectivity and Transmittance through an interface between 2 media for the light going from the medium 2 (refractive index n_2) to the medium 1 (refractive index n_1)

Using the relations $f_{21} = f_{12}^{-1}$ and $n_{21} = n_{12}^{-1}$, it can easily be shown that Table 2 is equivalent to Table 3.

Intensity for $n : 2 \rightarrow 1$	Reflectivity R_{12}	Transmittance T_{12}
s -polarization	$\left(-\frac{1-n_{21}f_{21}}{1+n_{21}f_{21}}\right)^2$	$n_{21}f_{21} \cdot \left(\frac{2}{1+n_{21}f_{21}}\right)^2$
p -polarization	$\left(-\frac{n_{21}-f_{21}}{n_{21}+f_{21}}\right)^2$	$n_{21}f_{21} \cdot \left(\frac{2}{n_{21}+f_{21}}\right)^2$

Table 3: Reflectivity and Transmittance through an interface between 2 media for the light going from the medium 2 (refractive index n_2) to the medium 1 (refractive index n_1) using n_{21} and f_{21}

This is identical to the reflectivity and transmittance when n is changing like $1 \rightarrow 2$, as shown in Table 1. The only difference consists in the different amplitude of the reflectivity, where the negative sign denotes the different behavior of a phase of an electromagnetic wave when passing from n_1 to n_2 with $n_1 < n_2$ as compared to $n_1 > n_2$. There is a phase shift of π . This however has no influence on the intensity or power/energy and therefore no influence on the reflectivity either. This important property should be noted:

For both $n : 1 \rightarrow 2$ and $n : 2 \rightarrow 1$, the following relations apply: $R_{12} = R_{21}$ and $T_{12} = T_{21}$ for both directions of polarization! These depend only on the relative difference between the refraction indices.

In a simplified approach, unpolarized light can be viewed as consisting in equal amounts of both directions of polarization (the s -direction and p -direction). In this case, the following relation is found:

$$I_e = (R_{21} + T_{21}) = \frac{1}{2}(R_{p21} + R_{s21}) \cdot I_e = \frac{1}{2}(T_{p21} + T_{s21}) \cdot I_e \quad (11)$$

For the special case of an orthogonal incidence $\alpha = 0$ it follows that $f_{21} = f_{12} = 1$. In this case the reflectivity and transmittance are identical for both polarizations and the Fresnel equations can be simplified to:

Intensity	Reflectivity R_{12}	Transmittance T_{12}
Any polarization	$\left(\frac{1-n_{21}}{1+n_{21}}\right)^2 = \left(\frac{n_2-n_1}{n_2+n_1}\right)^2$	$n_{21} \cdot \left(\frac{2}{1+n_{21}}\right)^2$

Table 4: Reflectivity and transmittance for $\alpha = 0$

1.4. Transmittance and Reflectivity of a Glass Plate

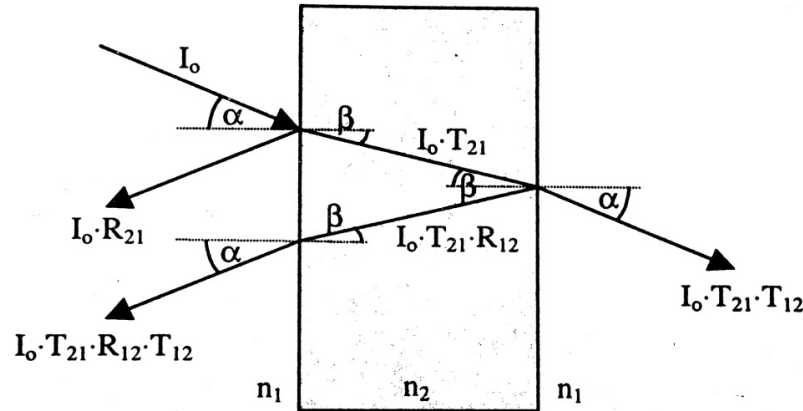


Figure 6: The basic reflections and transmissions of a glass plate.

The Fresnel equations can now be used to calculate the transmittance and reflectivity of a glass plate. For the calculations, absorption will not be taken into account. A glass plate consists of 2 phase transitions: air-glass (index 21) and glass-air (index 12). During each of these transitions a reflection will take place. In order to calculate interference one must also take a phase shift of π into account which occurs on a phase transition from an optically thin to an optically thick medium. Neglecting the interference, we can neglect the phase shift for now.

As a first approximation, multiple reflections will not be taken into account. Let's consider first the reflections shown in Figure 6. As defined, the incident beam is at an angle α to the perpendicular to the boundary area. Then, the coherent beam within medium 2 will have a diffraction angle of β and an output beam angle of α . The reflected beam at the phase boundary 2->1 also has an angle α to the perpendicular to the boundary area. In the described case the reflectivity and transmittance on both sides of the phase transition is equal. For the total amounts it leads to:

$$T_{tot} = T_{21} T_{12} = T_{21}^2$$

$$R_{tot} = R_{21} + T_{21} T_{12} R_{12} = R_{21} (1 + T_{21}^2)$$

Inserting the equations of Table 1, it results in the following equations for the polarization of a glass plate:

Reflectivity	
<u>perpendicular polarization</u>	$\left(\frac{1-n_{21}*f_{21}}{1+n_{21}*f_{21}}\right)^2 * \left\{1 + (n_{21} * f_{21})^2 * \left(\frac{2}{1+n_{21}*f_{21}}\right)^4\right\}$
<u>parallel polarization</u>	$\left(\frac{n_{21}-f_{21}}{n_{21}+f_{21}}\right)^2 * \left\{1 + (n_{21} * f_{21})^2 * \left(\frac{2}{n_{21}+f_{21}}\right)^4\right\}$
Transmittance	
<u>perpendicular polarization</u>	$(n_{21} * f_{21})^2 * \left(\frac{2}{1+n_{21}*f_{21}}\right)^4$
<u>parallel polarization</u>	$(n_{21} * f_{21})^2 * \left(\frac{2}{n_{21}+f_{21}}\right)^4$

Table 5: Reflectivity and transmittance of a single glass plate (first approximation)

The reflectivity and transmission for both polarizations are plotted in figure 7 for a glass plate with a refractive index of 1.52. The influence of the refractive index is shown in figure 8.

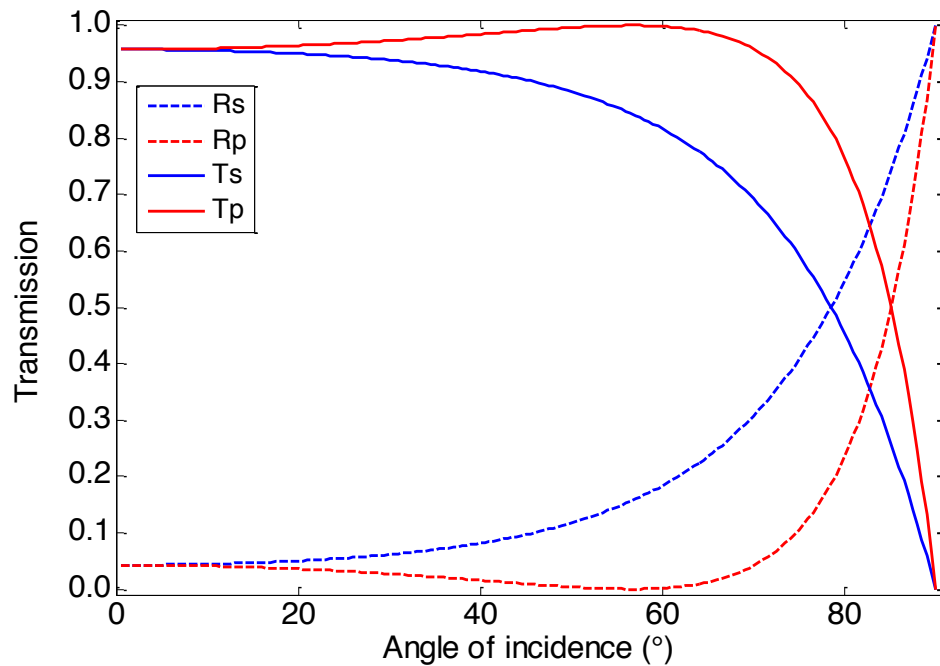


Figure 7: Reflectivity and transmission of a glass plate with $n=1.52$.

Brewster's angle (also known as the polarization angle) is an angle of incidence at which light with a particular polarization is perfectly transmitted through a surface, with no reflection. The angle at which this occurs is named after the Scottish physicist, Sir David Brewster (1781–1868).

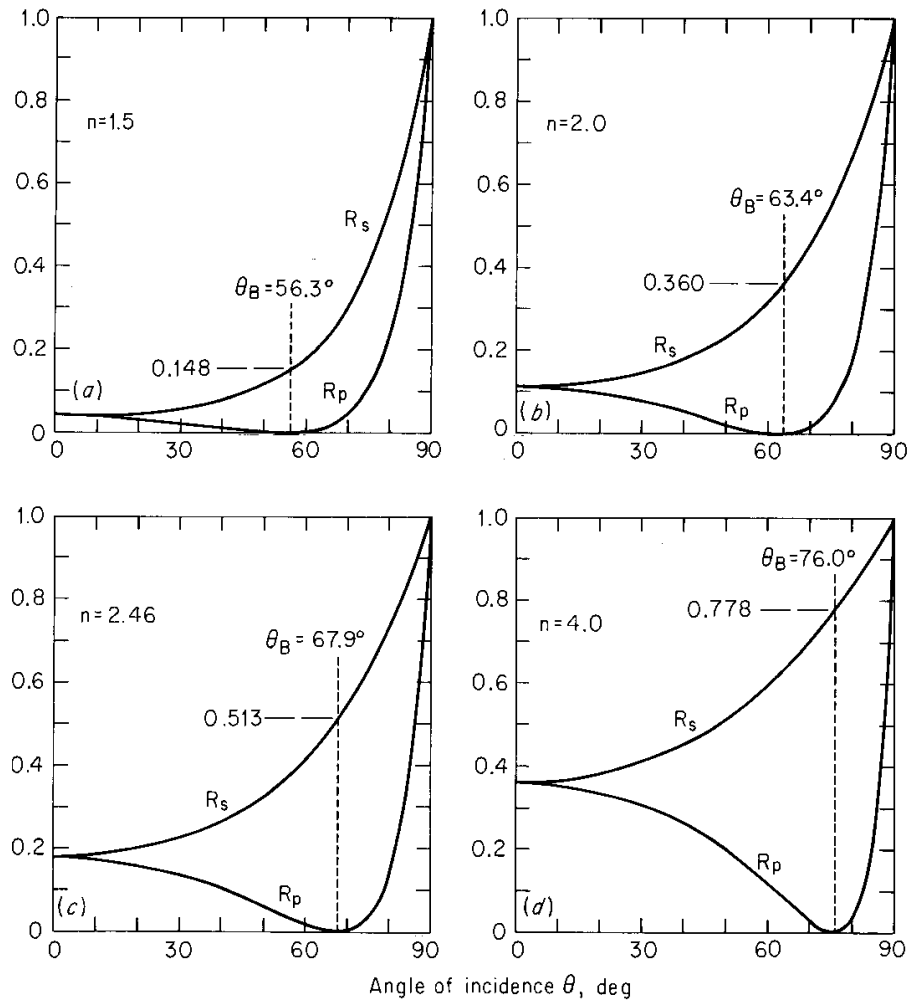


Figure 9: Reflectance of light polarized parallel R_p and perpendicular R_s to the plane of incidence from materials of different refractive index n as a function of angle of incidence : (a) $n = 1.5$ (normal glass) , (b) $n = 2.0$ (AgCl in infrared) , (c) $n = 2.46$ (Se in infrared) , and (d) $n = 4.0$ (Ge in infrared) . The Brewster angle ϑ_B (at which R_p goes to 0) and the magnitude of R_s at ϑ_B are also indicated. From [1].

If we look again at Figure 5 (left), with simple trigonometry the following condition can be expressed:

$$\theta_1 + \theta_2 = 90^\circ,$$

where θ_1 is the angle of incidence and θ_2 is the angle of refraction. Using Snell's law:

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2),$$

we can calculate the incident angle $\theta_1 = \theta_B$ at which no light is reflected:

$$n_1 \sin(\theta_B) = n_2 \sin(90 - \theta_B) = n_2 \cos(\theta_B).$$

Rearranging, we get:

$$\theta_B = \arctan\left(\frac{n_2}{n_1}\right),$$

where n_1 and n_2 are the refractive indices of the two media. This equation is known as Brewster's law. Note that, since all *p*-polarized light is refracted (i.e transmitted), any light reflected from the interface at this angle must be *s*-polarized. Although Brewster's angle is generally presented as a zero-reflection angle in textbooks from the late 1950s onwards, it truly is a polarizing angle.

1.5. Isotrop and Anisotrop Media/ Non-Linear Effects

The term optical isotropy is used for media which have the same optical properties for all spatial directions. This is normally true for gases and fluids. Solid states are crystalline. Due to the crystal structure the solid has mostly one (or two or three) crystalline principle axes which represent the symmetry axes of the crystal. Thus the optical properties of solids depend on the directions of the light propagation regarding the crystal axes; generally a solid would then be optically anisotropic. The symmetry axes are signaling a priority and are therefore called optical axes. Along an optical axis the refractive index stays constant, no double refraction is existent and the solid seems to be isotropic along that axis. Along all the other directions however the refractive index is depending on the angle to the optical axis. Other priority directions are set by one or two axes perpendicular to the optical axes. A beam spreading along one of these axes is called extraordinary ray. An extraordinary ray does not follow Snell's law. A beam spreading along the optical axes is called ordinary ray.

Additionally to a natural anisotropy or double refraction of a medium there also exists an induced double refraction, which is generated by an external electric or magnetic field. Then the induced changes in refractive indices are the following:

$\Delta n = K_P * \lambda * E$	Pockels effect
$\Delta n = K_K * \lambda * E^2$	Kerr effect
$\Delta n = K_F * \lambda * H^2$	Faraday effect

The Kerr effect could be further differentiated into an orientating Kerr effect and an electro-optic Kerr effect. In the first case molecules are orientated by an external field, in the second one the orbitals of the bound electrons are "orientated". The electro-optic effect is extremely fast with time constants of around $0.1 \text{ fs} = 10^{-16} \text{ s}$. This results in non linear effects that can react to an electric alternating field or an electro-magnetic wave. The beam of a HeNe-Laser(632.8 nm) has an optical frequency of 473 THz giving a periodic time of 2.1 fs.

2. Polarizers

2.1. Description of polarizers

Light from most natural sources tends to be randomly polarized. While there are a number of methods of converting it to linear polarization, only those that are commonly used in optical design will be covered. One method is reflection, since the amount of light reflected off a tilted surface is dependent on the orientation of the incident polarization and the normal to the surface. A geometry of particular interest is one in which the propagation direction of reflected and refracted rays at an interface are perpendicular to each other, as shown in Figure 8.

A linear polarizer is anything which when placed in an incident unpolarized beam produces a beam of light whose electric vector is vibrating primarily in one plane, with only a small component vibrating in the plane perpendicular to it (Annexes 5.1, 5.2 and 5.3). If a polarizer is placed in a plane-polarized beam and is rotated about an axis parallel to the beam direction, the transmittance T will vary between a maximum value T_1 and a minimum value T_2 according to the law:

$$T = (T_1 - T_2) \cos^2(\theta) + T_2$$

Although the quantities T_1 and T_2 are called the *principal transmittances*, in general $T_1 \gg T_2$; θ is the angle between the plane of the principal transmittance T_1 and the plane of vibration (of the electric vector) of the incident beam. If the polarizer is placed in a beam of unpolarized light, its transmittance is

$$T = 1/2 \cdot (T_1 + T_2)$$

so that a perfect polarizer would transmit only 50 percent of an incident unpolarized beam.

When two identical polarizers are placed in an unpolarized beam , the resulting transmittance will be

$$T_{||} = 1/2 \cdot (T_1^2 + T_2^2)$$

when their principal transmittance directions are parallel and will be

$$T_{\square} = T_1 \cdot T_2$$

when they are perpendicular. In general, if the directions of principal transmittance are inclined at an angle θ to each other, the transmittance of the pair will be

$$T_{\theta} = 1/2 (T_1^2 + T_2^2) \cdot \cos^2 \theta + T_1 \cdot T_2 \sin^2 \theta$$

The polarizing properties of a polarizer are generally defined in terms of its *degree of polarization* P

$$P = \frac{T_1 - T_2}{T_1 + T_2}$$

or its *extinction ratio* ρ_P

$$\rho_P = \frac{T_2}{T_1}$$

When one deals with nonnormal-incidence reflection polarizers, one generally writes P and ρ_P in terms of R_p and R_s , the reflectances of light polarized parallel and perpendicular to the plane of incidence, respectively . R_s can be equated to $T_{\perp}$ and R_p to $T_{\parallel}$, so that $P = (R_s - R_p) / (R_s + R_p)$ and $\rho_P = R_p / R_s$. If either ρ_P or P is known, the other can be deduced. If one is determining the degree of polarization or the extinction ratio of a polarizer, the ratio of $T_{||}$ to $T_{\square}$ can be measured for two identical polarizers in unpolarized light.

$$\frac{T_{\perp}}{T_{\parallel}} = \frac{T_1 \cdot T_2}{(T_1^2 + T_2^2)/2} \approx \frac{2T_2}{T_1} = 2\rho_P \quad \text{if } T_2^2 \ll T_1^2$$

If a perfect polarizer or a source of perfectly plane-polarized light is available, T_2 / T_1 can be determined directly by measuring the ratio of the minimum to the maximum transmittance of the polarizer. Other relations for two identical partial polarizers are given by West and Jones, as well as the transmittance $T_{\theta ab}$ of two dissimilar partial polarizers a and b whose principal axes are inclined at an angle θ with respect to each other. This latter expression is:

$$T_{\theta ab} = 1/2 (T_{1a} T_{1b} + T_{2a} T_{2b}) \cdot \cos^2 \theta + 1/2 (T_{1a} T_{2b} + T_{1b} T_{2a}) \sin^2 \theta$$

where the subscripts 1 and 2 refer to the principal transmittances, as before. Spectrophotometric measurements can involve polarizers and dichroic samples. Dichroic materials are those which absorb light polarized in one direction more strongly than light polarized at right angles to that direction . (Dichroic materials are to be distinguished from birefringent materials , which may have different refractive indices for the two electric vectors vibrating at right angles to each other but similar, usually negligible, absorption coefficients.). If we simplify this expression we obtain the Malus' law.

2.2. Polarization by Nonnormal-Incidence Reflection (Pile of Plates)

2.2.1. Introduction

One can construct a linear polarizer by stacking a number of glass slides (e.g., clean microscope slides) at Brewster's angle to the beam direction. As indicated in Figure 9, each interface rejects a small amount of light polarized perpendicular to the plane of incidence.

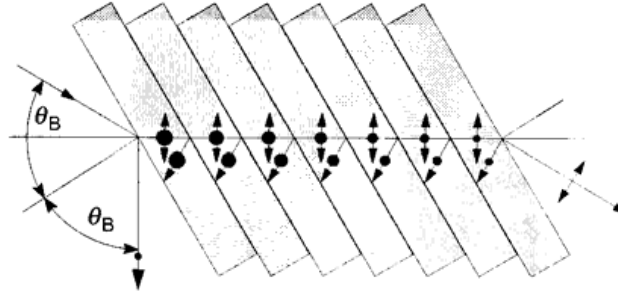


Figure 9 A “Pile of Plates” polarizer. This device working at Brewster angle, reflects some portion of the perpendicular polarization (here depicted as a dot, indicating an electric field vector perpendicular to the page) and transmits all parallel.

The “pile of plates” polarizer just described is somewhat bulky and tends to get dirty, reducing its efficiency. Plastic polarizing films are easier to use and mount. These films selectively absorb more of one polarization component and transmit more of the other.

2.2.2. Transmission through a pile of plates neglecting the internal reflections

To understand the concept, let's first neglect the internal reflection. See figure 6 in section 1.4. The difference in transmittance and reflectivity between s-polarization and p-polarization can be used to create a polarizer. The relative difference is equal to the ratio and therefore the contrast. For the reflectivity of a glass plate the relative difference is infinitely high at the Brewster's angle because there the reflected beam is totally polarized. This implies fully monochromatic light. For transmittance the contrast is increasing continuously with the incident angle, it never reaches a maximum at a certain angle. The simplest mathematical equations for the contrast derived from the equations of Table 5:

$$C_{trans} = \left(\frac{T_{tot,parallel}}{T_{tot,perpendicular}} \right)^N = \left(\frac{1 + n_{21} * f_{21}}{f_{21} * n_{21}} \right)^{4N}$$

The contrast is plotted for 1,2 and 4 glass plates in Figure 10. For a glass plate ($n=1.52$) and air ($n=1$). You will see, experimentally, that the first approximation is not sufficient and the multiple reflections have to be taken into accounts.

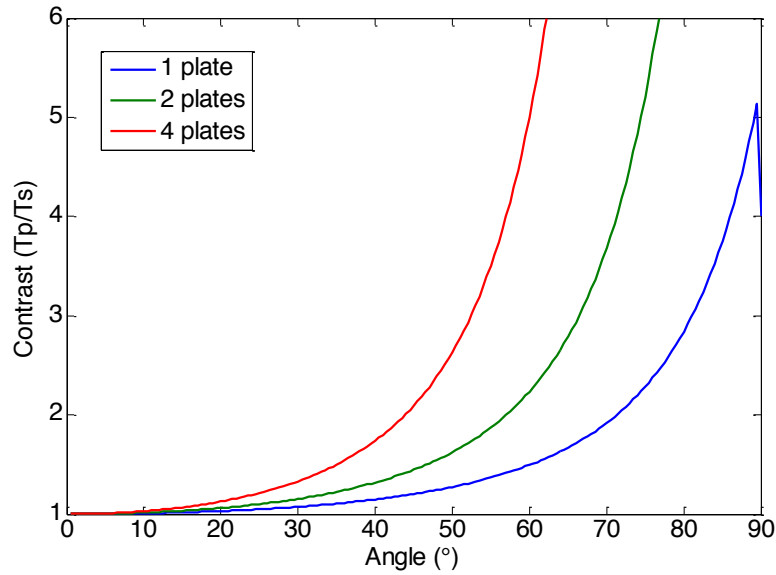


Figure 10: Transmittance contrast of one up to four glass plates.

2.2.3. Transmission through a Pile of Plates (full description)

Although in many cases multiple reflections within a plate degrade its polarizing properties, this is not true for Brewster angle reflection polarizers. For multiple reflections within a plane-parallel plate of material [1]:

$$R_{(s,p)plate} = 2 R_{s,p} / (1 + R_{s,p})$$

Multiple reflections have a minor effect on the extinction ratio but the increase in R_s is appreciable. The plate must have plane-parallel sides and be unbacked. We are also assuming that the plate is thick or nonuniform enough for interference effects within it to be neglected. All the preceding discussion applies only to nonabsorbing materials. If a small amount of absorption is present, R_p will have a minimum that is very close to zero and the material will still make a good reflection polarizer. However, if the extinction coefficient becomes appreciable, the minimum in R_p will increase and the polarizing efficiency will be degraded.

Table 5 (from [1]) lists the relations giving the s and p transmittances of the polarizers with various assumptions about multiple reflections, interference, absorption, etc. All these relations contain R_s and R_p , the reflectances at a single interface, which are given at the bottom of the table. At the Brewster angle, R_p at a single interface equals zero, and the transmittances of the plates can be expressed in terms of the refractive index of the material and the number of plates.

$\dagger \alpha = 4\pi k/(\lambda \cos \theta_1)$, $\gamma = 2\pi nd \cos \theta_1/\lambda$, θ_0 = angle of incidence, θ_1 = angle of refraction, n = refractive index = $(\sin \theta_0)/(\sin \theta_1)$, k = extinction coefficient, d = plate thickness, λ = wavelength.
 $\ddagger$ No multiple reflections between plates.
 $\S$ Multiple reflections between plates.
 $\parallel$ Also holds for coherent multiple reflections averaged over one period of $\sin^2 \gamma$.

The s and p transmission for 1,2 and 4 plates are plotted in Figure 11 and the contrasts are plotted for 1,2 and 4 glass plates in figure 12 for a glass plate index $n=1.52$. Multiple reflections have been taken into accounts, but interferences and absorption have been neglected.

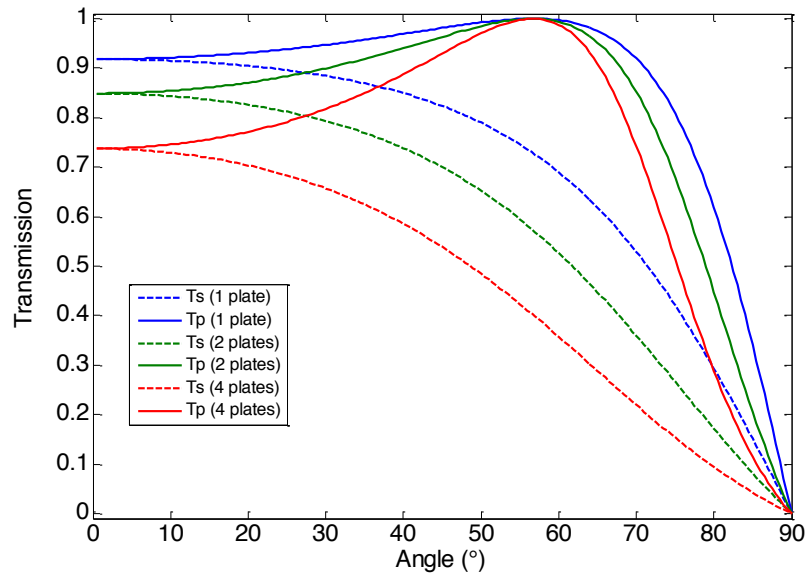


Figure 11: s and p transmission for 1,2 and 4 glass plates with $n=1.52$.

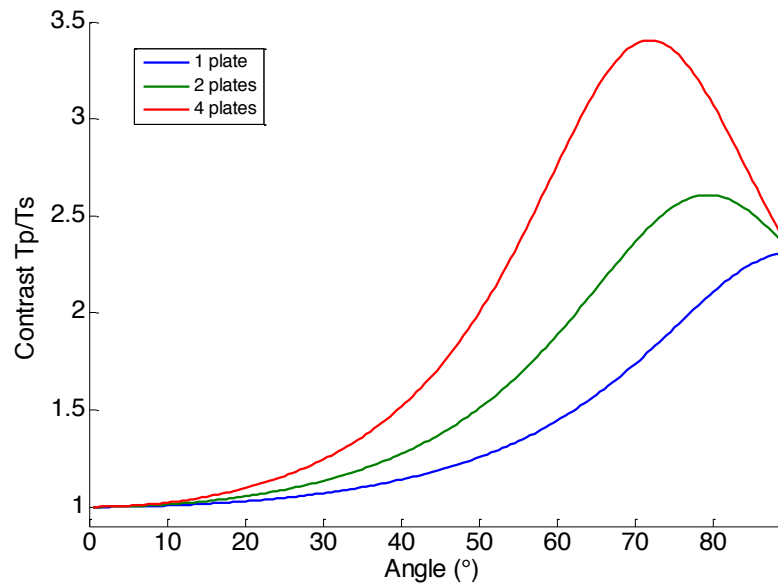


Figure 12: Contrast between s and p polarization for 1,2 and 4 glass plates with $n=1.52$.

3. Applications

3.1. Polarization of light as analyzer of material

In many cases polarizers are used to provide information about a material that produces, in some manner, a change in the form of polarized light passing through it. The standard configuration, shown in Figure 13, consists of a light source S , a polarizer P , the material M , another polarizer, called an analyzer A , and a detector D . Usually the polarizer is a linear polarizer, as is the analyzer. Sometimes, however, polarizers that produce other types of polarization are used. If the material M changes the polarization, it can be detected by the analyzer.

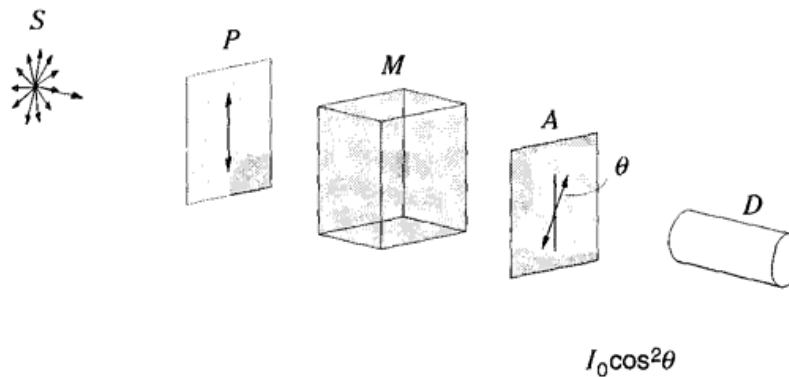


Figure 13: Setup using polarization of light as analyzer of material

Polarised light can be used to measure strain in photoelastic materials, such as glass and celluloid. These are materials that become birefringent when placed under mechanical stress. One application of this effect is in stress analysis. A celluloid model of a machine part, for example, is placed between a crossed polariser and analyser. The model is then placed under stress to simulate working conditions. Bright and dark fringes appear, with the fringe concentration highest where the stress is greatest. This sort of analysis gives important information in the design of mechanical parts and structures.

Polaroid sunglasses and camera lens filters are often used to reduce glare. In the ideal case, light reflected from a horizontal surface will be polarised in a horizontal plane as described earlier, so a Polaroid with a vertical transmission axis should prevent transmission completely. In practice, reflected light is only partially polarized, and is not always being reflected from a horizontal surface, so glare is only partially reduced by Polaroid sunglasses and filters.

Optically active materials can change the plane of polarization of a beam of light. This process comes about because of the molecular structure of these materials, and has been observed in crystalline materials such as quartz and organic (liquid) compounds such as sugar solutions. The degree of optical activity can be used to help determine the molecular structure of these compounds. In the technique known as saccharimetry, the angle of rotation of the plane of polarization is used as a measure of the concentration of a sugar solution. Polarized light is passed through an empty tube, and an analyzer on the other side of the tube is adjusted until no light is transmitted through it. The tube is then filled with

the solution, and the analyzer is adjusted until the transmission through it is again zero. The adjustment needed to return to zero transmission is the angle of rotation.

3.2. Liquid crystal displays (LCDs)

Perhaps the most common everyday use of optical activity is in liquid crystal displays (LCDs). A typical LCD on a digital watch or electronic calculator consists of a small cell of aligned crystals sandwiched between two transparent plates between a crossed polariser and analyser. This arrangement is shown schematically in Figure 14.

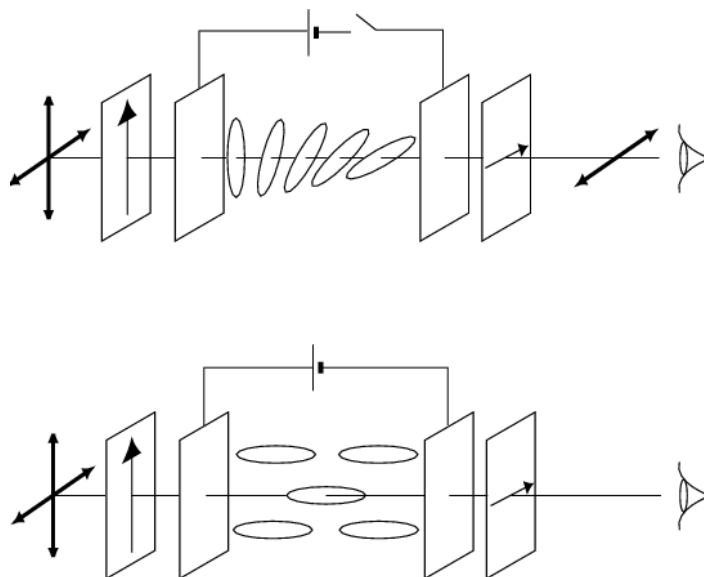


Figure 14 - LCD display with no field applied (top) and with a field applied (bottom)

When no electric field is applied across the cell, the liquid crystal molecules are arranged in a helical twist. The polarized light entering the cell has its polarization angle changed as it travels through the cell, and emerges polarized parallel to the transmission axis of the analyzer. The cell appears light in color, and is thus indistinguishable from the background. When a field is applied, the liquid crystal molecules align in the same direction (the direction of the electric field), and do not change the polarization of the light. The emerging light remains polarized perpendicular to the analyzer transmission axis. This light is not transmitted by the analyzer, and so the cell looks dark. In this case, a black segment is seen against a lighter background.

3.3. Pockels cells and Electro-Optic Modulators (EOM)

A Pockels cell is a device consisting of an electro-optic crystal (with some electrodes attached to it) through which a light beam can propagate [11]. The phase delay in the crystal ($\rightarrow$ Pockels effect) can be modulated by applying a variable electric voltage. The Pockels cell thus acts as a voltage-controlled waveplate. Pockels cells are the basic components of electro-optic modulators, used e.g. for Q switching lasers. The majority of electro-optic devices are based on the Pockels effect. Structurally isotropic materials, including all gases, liquids, and amorphous solids such as glass, show no Pockels effect because they are centrosymmetric. The index changes induced by the Pockels effect can be utilized to construct a variety of electro-optic modulators, in either bulk or waveguide structures. An electro-optically induced rotation of principal axes is not required for the functioning of an electro-optic modulator though it often accompanies the index changes.

Pockels cells can have two different geometries concerning the direction of the applied electric field:

- Longitudinal devices have the electric field in the direction of the light beam, which passes through holes in the electrodes. Large apertures can easily be realized, as the required drive voltage is basically independent of the aperture. The electrodes can be metallic rings (Figure 1, left) or transparent layers on the end faces (right) with metallic contacts.

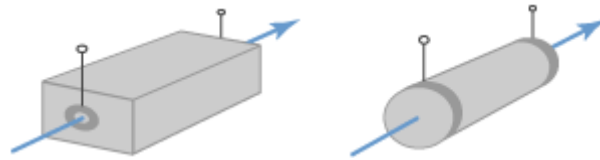


Figure 15: Pockels cells with longitudinal electric field.

- Transverse devices have the electric field perpendicular to the light beam. The field is applied through electrodes at the sides of the crystal. For small apertures, they can have lower switching voltages.

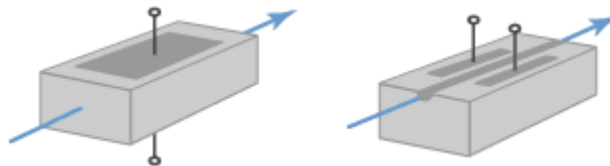


Figure 16: Pockels cells with transverse electric field.

On the left is a bulk modulator and on the right a waveguide modulator.

Common nonlinear crystal materials for Pockels cells are potassium di-deuterium phosphate ($\text{KD}^*\text{P} = \text{DKDP}$), potassium titanyl phosphate (KTP), β -barium borate (BBO) (the latter for higher average powers and/or higher switching frequencies), lithium niobate (LiNbO_3), lithium tantalate (LiTaO_3), and ammonium dihydrogen phosphate ($\text{NH}_4\text{H}_2\text{PO}_4$, ADP).

An electro-optic modulator (EOM) (or electrooptic modulator) is a device which can be used for controlling the power, phase or polarization of a laser beam with an electrical control signal. It typically contains one or two Pockels cells, and possibly additional optical elements such as polarizers. Different types of Pockels cells. The principle of operation is based on the linear electro-optic effect (also called the Pockels effect), i.e., the modification of the refractive index of a nonlinear crystal by an electric field in proportion to the field strength.

Combined with other optical elements, in particular with polarizers, Pockels cells can be used for other kinds of modulation. In particular, an amplitude modulator (Figure 17) is based on a Pockels cell for modifying the polarization state and a polarizer for subsequently converting this into a change in transmitted optical amplitude and power.

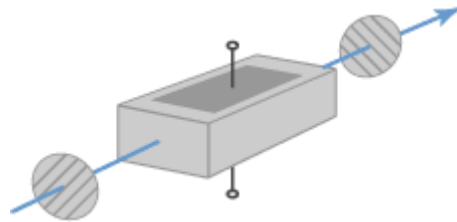


Figure 17: Electro-optic amplitude modulator, containing a Pockels cell between two polarizers.

An alternative technical approach is to use an electro-optic phase modulator in one arm of a Mach–Zehnder interferometer in order to obtain amplitude modulation. This principle is often used in integrated optics (for photonic integrated circuits), where the required phase stability is much more easily achieved than with bulk optical elements.

Optical switches are modulators where the transmission is either switched on or off, rather than varied gradually. Such a switch can be used, e.g., as a pulse picker, selecting certain pulses from a train of ultrashort pulses, or in cavity-dumped lasers (with an EOM as cavity dumper) and regenerative amplifiers.

4. Experimental Studies

Left setup:

- 1 un-polarized He-Ne Laser
- 1 optical bench with a rotation arm
- 2 amplified photo detectors (PD)
- 2 multi-meters (MM) and cables
- 2 lenses
- A glass holder and 4 glass plates
- 2 graduated polarizer (P) (to be used in the right setup as well)
- 2 small polarizer with a potentiometer (P)
- A 15V voltage source
- 2 Irises
- A computer with an analogue to digital converter to measure a voltage input on (two channels) and plot them on an X-Y graph using a Labview program

Right setup:

- 1 polarized He-Ne Laser
- 1 Pockels Cell (PC)
- 1 optical bench
- 1 photo detector (PD)
- 1 oscilloscope
- 2 graduated polarizer (P)
- A roll of frosted adhesive tape
- Please bring your digital camera

4.1. Demonstration of the Malus' Law

On the left setup, the student will first analyze the polarization states of the laser (supposed to be unpolarized) using a single polarizer. Note that the laser needs to be warm to have a time-independent stable intensity. One polarizer has a potentiometer to transfer the polarization angle to a voltage using the voltage source. This voltage output will be recorded via the analogue to digital converter by the Computer. As a second voltage signal the photo detector will be recorded by the computer. Both signals correspond to a certain polarization angle and laser light intensity, respectively and will be plotted with a Labview program. Note that the laser is not perfectly unpolarized. Directly after the laser output a second fixed polarizer P2 will be placed to create a defined laser light polarization. By turning the first polarizer P1, the student will measure Malus' law as a function of polarization angle of P1 and corresponding laser light intensity. A scan of the plots will be added into the report.

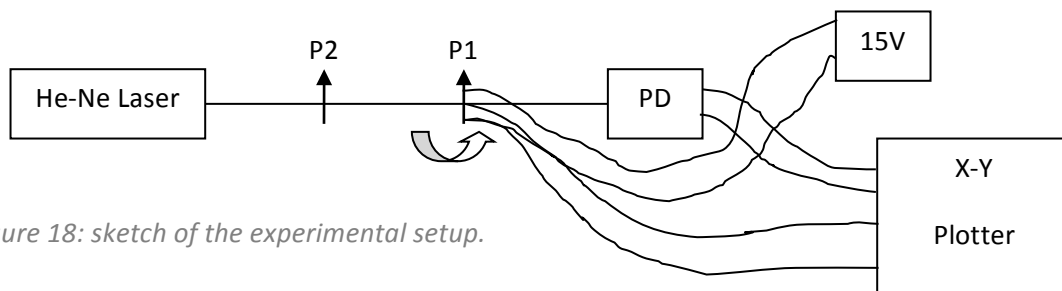


Figure 18: sketch of the experimental setup.

4.2. s and p transmission and reflection through and on a glass plate

The student will measure the intensity of the reflected and transmitted beam from a glass plate for both s and p polarization using the amplifying detectors. In this experiment we will demonstrate the equations from Table 5 and should get the similar curves as in Figure 7. Note that the photo detectors have a ground level that needs to be subtracted from the signal and that the laser is not extremely stable. Rigorous measurements imply to measure for each angle and polarization the intensity of the laser after the polarizer and without glass plate (I_0), the transmitted and reflected powers. In the report an error analysis will be done. An iris in front of detector 2 helps to improve the angular resolution. Avoid stray light to minimize the systematic error. A third detector PD3 together with an additional non-polarizing beam splitter after P1 could be an alternative possibility to record the laser intensity (I_0) throughout the experiment. If this will be done it is very important to calibrate the third detector with PD1 and PD2. Furthermore it is important the calibration of PD1 and PD2, i.e. what is the voltage on both detectors when they measure the same signal.

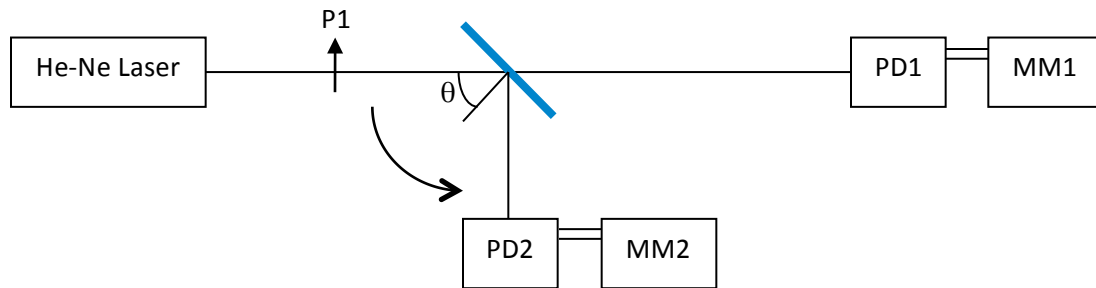


Figure 19: sketch of the experimental setup.

4.3. A pile of glass plates polarizer

The student should measure for 1, 2 and 4 glass plate(s) the contrast in the transmitted polarization. In this experiment we will demonstrate the equations from Table 6 and should get the similar curves as in Figure 11 and 12. Note that the photo detectors have a ground level and that the laser is not extremely stable. Rigorous measurements imply to measure for each angle and polarization the intensity of the laser with glass plates (I_0), the transmitted and reflected powers. In the report an error analysis will be done. Note that at large angles, part of the light might be blocked by mounts.

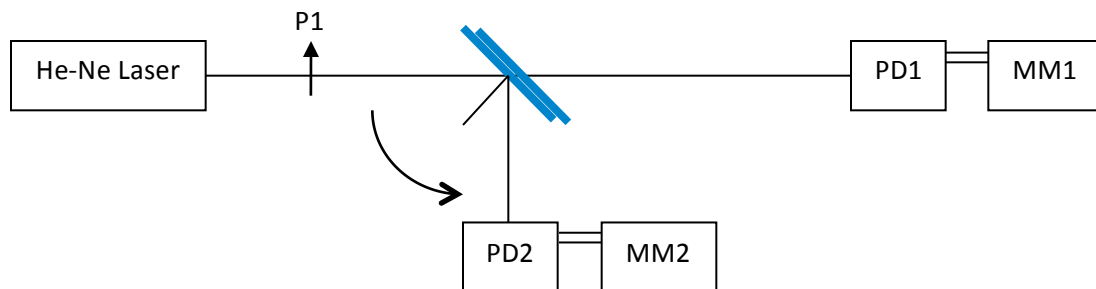


Figure 20: sketch of the experimental setup

4.4. Construction of an EOM using a Pockels cell.

On the right setup, we will observe the fast modulation of light using a Pockels cell, 2 polarizers and a high voltage source.

First the pockels cells needs to be aligned. Follow the instruction in Section 5.5. and take few images of the isogyre pattern [12] with a digital camera. The pockels cell will be directly attached to the frequency generator.

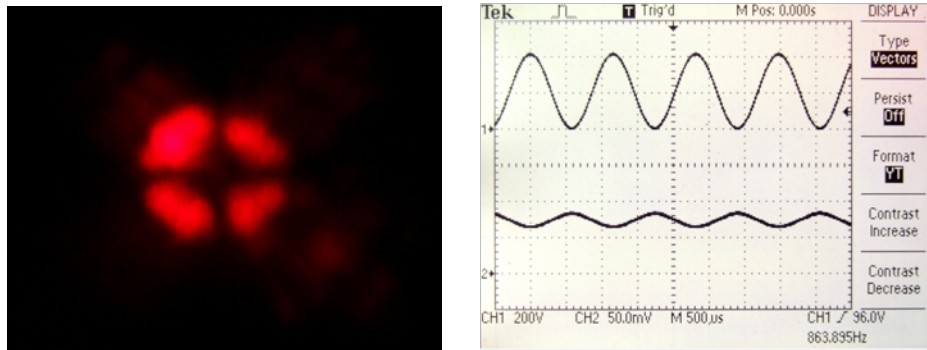


Figure 21: Example of an isogyre pattern [12] and a screen shot of the oscilloscope

The cross pattern is formed because, as the light passes through the structure, there is a rotation of the plane-polarized light in all planes except those parallel to the polarizer and analyzer axes, and the extent of this rotation varies depending on the path that the light takes. Consequently, some light is able to pass through the second polarizer (analyzer), and there are variations in the light intensity within the quadrants that separate the isogyres. It enhanced with slightly divergent beams.

After aligning the Pockels cell connect the frequency generator output and the photo detector to the oscilloscope. Record for 10 different frequencies the peak to peak values of the sinusoidal photo detector signal and plot them in the report. As the frequencies are increased the peak to peak values decrease. Explain why this is the case.

4.5. Measurement of sugar concentration

Another use of polarized light is to measure the sugar content of various syrups in the candy making industry. The basis for this measurement is the fact that sugars are optically active materials, causing a rotation of the plane of polarization of light passing through the thickness of the sample through which the light travels. Thus by constructing a device with a standard sample length and knowing the rotation constant of the sugar, the concentration of an unknown sugared liquid can be determined. For example colorless syrup has a rotational constant for visible light that is easily measured.

Align the glassware such that the laser light passes through the glass but not the glucose solution and rotate the analyzer such that we don't have any light on the detector (cross polarization condition) and record the angle α_1 on the analyzer. Shift the glassware upwards such that the beam passes the glucose solution. In the glucose solution, the light will be rotated and the laser light will be partly transmitted by the analyzer. Now rotate the analyzer to obtain the cross polarization condition and record the angle α_2 under which this condition occurs. The difference between these two angles $|\alpha_1 - \alpha_2|$ gives the angle of rotation and the direction of rotation caused by the glucose.

From the angular rotation measurements, using the specific rotation of glucose ($\alpha = 45.62$ degrees/cm/(g/mL) for a wavelength of 633 nm) calculate the concentration of glucose in solution, where θ is the AVERAGE measured angle of rotation, L is the path length through the solution, and C is the concentration (g/mL) of the glucose solution.

$$\theta = \alpha \cdot L \cdot C$$

Based on a minimum measurable rotation of about 0.5 degrees, what is the smallest concentration of glucose you would be able measure with this experimental set up? Explain how you estimate this limit.

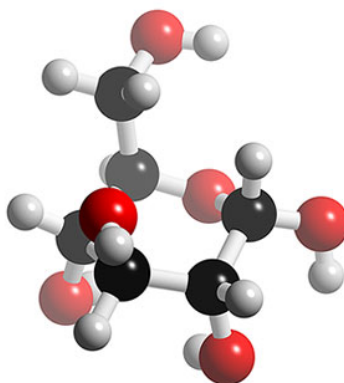


Figure 22: Molecule of Glucose ($C_6H_{12}O_6$)

5. Annexes

5.1. Polaroid (3M)® Material

Polaroid is the trade name for the most commonly used dichroic material. It selectively absorbs light from one plane, typically transmitting less than 1% through a sheet of polaroid. It may transmit more than 80% of light in the perpendicular plane. The word "polaroid" usually refers to polaroid H-sheet, which is a sheet of iodine-impregnated polyvinyl alcohol. A sheet of polyvinyl alcohol is heated and stretched in one direction while softened, which has the effect of aligning the long polymeric molecules in the direction of stretch. When dipped in iodine, the iodine atoms attach themselves to the aligned chains. The iodine atoms provide electrons which can move easily along the aligned chains, but not perpendicular to them. Light waves with electric fields parallel to these chains are strongly absorbed because of the dissipative effects of the electron motion in the chains. The direction perpendicular to the polyvinyl alcohol chains is the "pass" direction since the electrons cannot move freely to absorb energy.

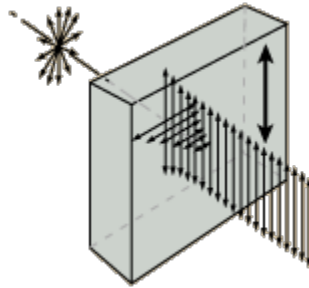


Figure 23: polarization in a Polaroid material

5.2. Glan–Foucault prism

A Glan–Foucault prism deflects p-polarized light, transmitting the s-polarized component. The optical axis of the prism material is perpendicular to the plane of the diagram. A Glan–Foucault prism (also called a Glan–air prism) is a type of prism which is used as a polarizer. It is similar in construction to a Glan–Thompson prism, except that two right-angled calcite prisms are spaced with an air-gap instead of being cemented together. Total internal reflection of p-polarized light at the air gap means that only s-polarized light is transmitted straight through the prism.

Compared to the Glan–Thompson prism, the Glan–Foucault has a narrower acceptance angle over which it will work, but because it uses an air-gap rather than cement, much higher irradiances can be used without damage. The prism can thus be used with laser beams. The prism is also shorter (for a given usable aperture) than the Glan–Thompson design, and the deflection angle of the rejected beam can be made close to 90° , which is sometimes useful. Glan–Foucault prisms are not typically used as polarizing beamsplitters because while the transmitted beam is 100% polarized, the reflected beam is not. The Glan–Taylor prism is very similar, except that the crystal axes and transmitted polarization direction are orthogonal to the Glan–Foucault design. This yields higher transmission, and better

polarization of the reflected light. Calcite Glan–Foucault prisms are now rarely used, having been mostly replaced by Glan–Taylor polarizers and other more recent designs.

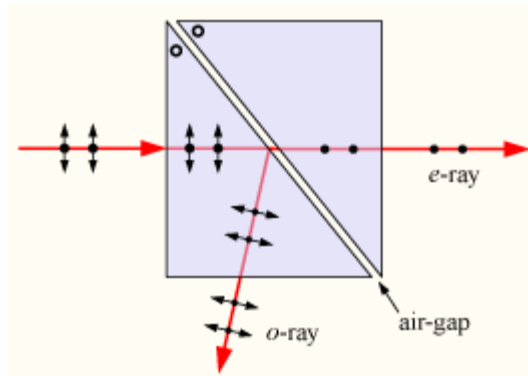


Figure 24: polarization in a Glan–Foucault prism

5.3. Nicol prism

A Nicol prism is a type of polarizer, an optical device used to generate a beam of polarized light. It was the first type of polarizing prism to be invented, in 1828 by William Nicol (1770-1851) of Edinburgh. It consists of a rhombohedral crystal of calcite (Iceland spar) that has been cut at a 68° angle, split diagonally, and then joined again using Canada balsam. Canada blasm is a transparent liquid.

Unpolarized light enters one end of the crystal and is split into two polarized rays by birefringence. One of these rays (the ordinary or o-ray) experiences a refractive index of $n_o = 1.658$ and at the balsam layer (refractive index $n = 1.55$) undergoes total internal reflection at the interface, and is reflected to the side of the prism. The other ray (the extraordinary or e-ray) experiences a lower refractive index ($n_e = 1.486$), is not reflected at the interface, and leaves through the second half of the prism as plane polarized light.

Nicol prisms were once widely used in microscopy and polarimetry, and the term "crossed Nicols" (abbreviated as XN) is still used to refer to observation of a sample between orthogonally orientated polarizers. In most instruments, however, Nicol prisms have been supplanted by other types of polarizers such as Polaroid sheets and Glan–Thompson prisms.

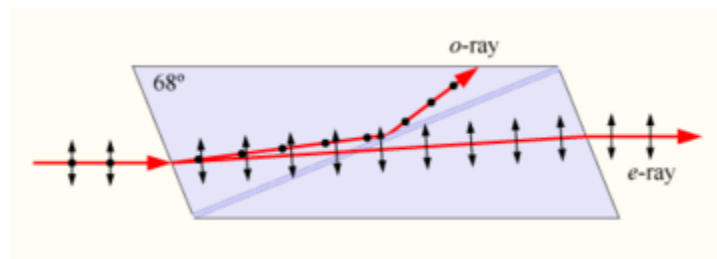


Figure 25: polarization in a Nicol prism

5.4. Optical and electrical risks

- Work with low power lasers < 5 mW
- Do not look directly in laser beam
- Remove your watch and other jewelry on your hands to avoid any reflection of the laser beam
- Avoid eye exposure to all kind of reflection of laser beam
- Never try to check laser work status by looking inside of the laser tube
- He-Ne laser has 4 to 10 kV voltage in the cable between laser tube and power supply.
- Never remove the cable from power supply!
- Switch off the power supply of the Pockels cell before changing the BNC cables

5.5. Procedure to align a Pockels cell

The following procedure has been shown to be most reliable for obtaining optimum alignment. The object is to centre the laser beam in the device apertures and then generate an optical pattern which accurately locates the optical axis of the crystal with respect to the laser beam. The procedure will probably require several adjustments of pitch, azimuth and translation to optimize the alignment but it will provide positive and visual confirmation of the alignment. The basic alignment configuration is shown in Figure 26.

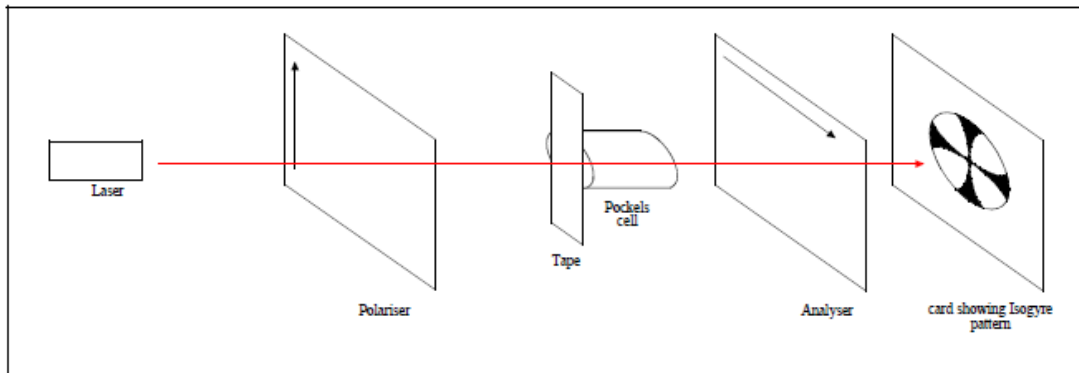


Figure 26: Modulator between Crossed Polarizers

1. Remove any polarizers used to polarize the beam entering the device. If the laser is already polarized it does not effect this procedure, however, the plane of polarization must be aligned (eyeball accuracy is usually good enough) with the marks or terminals on the Pockels cell. Position the device in the HeNe laser beam so that the beam is centered and passes through the apertures without touching the aperture edges.
2. Place a light colored matt finish card in the path of the beam at a distance of between 40-60 cm from the exit aperture of the Pockels cell. If the device is located within a laser cavity, the card should be placed against the laser rod holder and a small hole made in the card to locate the centre of the rod aperture. Mark the beam location on the card with a circle or dot and leave the card in place.
3. Place the input polarizer in the beam with its polarizing axis aligned to the mark on the Pockels cell aperture plate. If the laser rod produces a polarized beam (as with a ruby rod) the polarizer must be

aligned to the rod polarization direction. It is assumed that the polarizer does not deviate the beam angularly. Place the output polarizer (analyzer) at the output side of the device and insure that its polarizing axis is rotated 90° from that of the input polarizer.

4. Place a strip of frosted adhesive tape (Scotch Magic Mending Tape No. 810 or similar material) over the device entrance aperture. Gently press the tape in place but do not allow it to touch the window surface. A lightly frosted glass plate will provide the same scattering but must be nearly in contact with the entrance aperture. The actual measurement is usually made in a darkened room after basic alignment and adjustments are completed. In most instances, the pattern to be viewed will be difficult to see in normal room lighting. When the HeNe beam propagates through the optical train, a pattern, or some part of it, will be projected on the card. This is called an isogyre pattern [12] and is illustrated in Figure 27 a and 27b below.

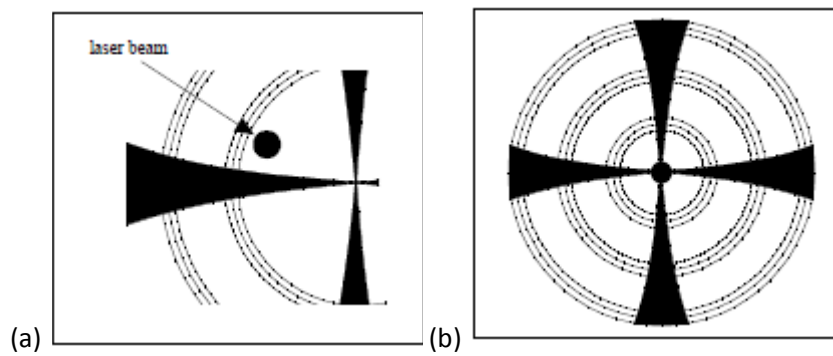


Figure 27: (a) Isogyre Pattern Off Centre (b) Isogyre Pattern Centered

If the optic axis of the crystal is not parallel to the path of the laser beam, the isogyre pattern [12] will be off-centre and the device must be moved in pitch and azimuth. When the isogyre is centered over the circle or dot or hole in the card this indicates that the device is well aligned. After making any positional adjustments, the beam position relative to the device aperture stops must be confirmed. The beam must still pass through both apertures without vignetting and with adequate clearance. If it does not, employ horizontal and vertical translation until clearance is confirmed. If the figure is not in the form of a cross, then the polarizers are not rotationally aligned to the faceplate mark or at 90° to each other.

Once the cross of the isogyre is centered, the polarizer can be rotated slightly to maximize the darkness of the centre of the cross. After this is done, the device is nulled with respect to the crossed polarizers for best contrast ratio and it is ready for operation.

6. References

1. Jean M. Bennett, Chapter 5 of "Handbook of Optics", McGraw-Hill Professional press (2009)
2. Jia-Ming Liu, "Photonic Devices" Cambridge University Press, (2005)
3. R. Goldstein, "Electro-Optic Devices in Review", Lasers & Applications, April, 1986
4. N. Stelmakh, Spring 2005 EE4344 "Polarization of Light"
5. Marcus Dülk (April 2000) Physikalisches Praktikum für Vorgerückte (VP) "Lichtpolarisation"
6. <http://scholar.hw.ac.uk/site/physics/topic1.asp?outline=no>
7. http://en.wikipedia.org/wiki/Snell%27s_law
8. <http://en.wikipedia.org/wiki/Polarization>
9. <http://en.wikipedia.org/wiki/Polarizer>
10. <http://hyperphysics.phy-astr.gsu.edu/hbase/ligcon.html#c1>
11. <http://www.rp-photonics.com/>
12. B. Pierscionek and J. Lekner, "Polarization patterns in refracting structures," Journal of the Optical Society of America **A**, Volume 14, pages 676-681, (1997).