Parametric Resonance, VP

Contents

1 Introduction

Harmonic oscillators are fundamental concepts in both classical and quantum mechanics, a (mathematical) pendulum being a simple example that can easily be calculated analytically for small displacements.

Generalising the concept a little by e.g., allowing large displacements increases the complexity of the problem remarkably (remember the assumption $\sin(\phi) \approx \phi$, which now no longer holds). For example, one has to take into account terms of higher order like the restoring force αx^3 or nonlinear damping $\eta x^2 \dot{x}$.
More involved systems than a mathematical pendulum make it further necessary that the parameters (e.g. length and mass of the pendulum, spring constant) are not constant over time but change with a certain frequency.

This allows to describe a wide range of new problems, for example a ship experiencing parametric rolling.

On the other hand, **parametric oscillators**, i.e. oscillators with variable parameters, play an important role in many areas of modern experimental physics as they offer interesting properties which can be exploited. Examples are not only parametric driving (which you will do in this experiment) but range from classical low-noise signal amplification to quantum parametric squeezing.

Besides their importance in modern physics, behaviour and applications of parametric resonators are still an active field of research, as the non-linear properties can lead to rich physics under different settings. Understanding the underlying physics is of paramount importance.

This experiment is a simplification of the experiment done by A. Leuch *et al.* at ETH, see [1]. The student lab setup allows, in principle, to study many phenomea ranging from nonlinear dynamics to parametric symmetry breaking, parametric amplification, and noise squeezing. The goal is to understand the main behaviours of non-linear parametric oscillators by observing many basic phenomena under different conditions, analogously as they were observed in [1]. By doing so, one also grasps on the techniques used for measuring small signals in the presence of significant noise.

2 Nonlinear Resonators

We limit ourselves to a nonlinear resonator of Duffing type and consider terms up to third order. The **equation of motion** is then given by, see [2],

$$\ddot{x} + \Gamma\,\dot{x} + \eta\,\dot{x}x^2 + \alpha\,x^3 + \Omega^2(t)\,x = \frac{F(t)}{M}, \tag{1}$$

where $\Omega(t)$ is the (time-dependent) natural resonance frequency (for small amplitudes), $F(t)$ is a driving force, M is the mass of the oscillator, and Γ, η and α are the coefficients of a nonlinear spring effect and of nonlinear damping, respectively. Here, the parametric modulation is represented via time dependence of Ω.
For small amplitude oscillations, (1) reduces to the equation of motion of a damped harmonic oscillator.

The **quality factor** (or Q-factor) of such an oscillator is related to the ratio of stored and dissipated energy in the oscillator, and for the harmonic oscillator determines the shape of the Lorentzian response to a driving force. It is given by

$$Q = \frac{M\omega_0}{\Gamma} = \frac{1}{2\zeta}, \tag{2}$$

where ω_0 is the natural resonance frequency and ζ is the damping ratio.

If oscillation amplitudes are large, nonlinear terms in (1) cannot be neglected and lead to anharmonic behaviour.

2.1 The Anharmonic Resonator

We consider now a special case of (1), where the driving force is of the form $F(t) = F_0\cos(\omega t)$, and $\Omega(t) = \omega_0 = \text{const.}$, i.e. there is no parametric modulation.
For large oscillations, the **Duffing term** α is responsible for the main damping factor of the oscillator. The **resonance frequency** ω_{res} becomes dependent on α and the **motional amplitude** x_0 of the oscillation; it is shifted from the natural resonance frequency ω_0 according to

$$\omega_{\text{res}} = \omega_0 + \frac{3}{8}\frac{\alpha}{M\omega_0}x_0^2 \tag{3}$$

The response of the resonator to the driving frequency ω becomes **non-Lorentzian**, meaning that the shape of the resonance peak is distinctly different from that of a harmonic oscillator.

2.2 The Parametric Resonator

If $\Omega(t)$ is not constant, the resonator is called a **parametric resonator**. Parametric resonance can be achieved by modulating the spring constant at twice the frequency of the driving force close to the resonance frequency, i.e. of the form

$$\Omega^2(t) = \omega_0^2(1 + \lambda \cos(2\omega t)), \quad F(t) = F_0 \cos(\omega t + \phi), \tag{4}$$

where $\omega \approx \omega_{\text{res}}$, λ is the **modulation depth**, and ϕ is a phase difference.

The oscillation amplitude experiences a **gain** relative to the case $\lambda = 0$ given by

$$G = \left| e^{-\pi/4} \left(\frac{\cos(\phi + \pi/4)}{1 - \lambda Q/2} + i \frac{\sin(\phi + \pi/4)}{1 + \lambda Q/2} \right) \right|. \tag{5}$$

There is a degeneracy for the phase ϕ, since $G(\phi) = G(\phi + \pi)$. For $\phi = -\pi/4$, the gain is maximal, while for $\phi = \pi/4$, G is smaller than 1, so-called squeezing.

At the **instability threshold** $\lambda_{\text{th}} = 2/Q$, the gain G diverges, and there are no stable solutions of the linear oscillator. The term from nonlinear spring constant becomes more impotant and thus saturate the diverging ampltude. The resonator can thus undergo large oscillations even without an external driving force ($F_0 = 0$), and is damped mainly by the Duffing term in (1).
Beyond the instability threshold, the oscillator can assume multiple amplitudes for a given driving frequency. Which amplitude is reached depends on the previous state of the oscillator, a so-called **hysteresis**.
If the resonator is not settled in one of these phases, the amplitude will behave differently, and no large scale oscillation will be observed.

The **instability region** over *omega* where oscillation is possible depends on the modulation depth λ. Outside this region, the phase of the resonator is not well-determined, behaves erratically, and the oscillation is not maintained. This instability region is known as the **Arnold's Tongue**.

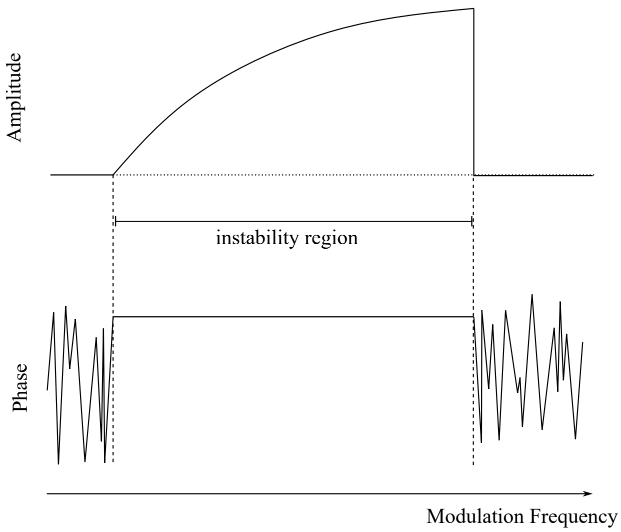

Figure 1: Schematic illustration of the oscillation amplitude and phase (relative to a reference signal), as a function of the modulation frequency ω.

3 Lock-in Amplifiers

This experiment aims at measuring periodic signals with significant noise. To solve this task, a lock-in amplifier is used.

In order to understand how a lock-in amplifier it is helpful to first recall the following identity

$$\sin(a)\sin(b) = \frac{1}{2}\big(\cos(a-b) - \cos(a+b)\big) \tag{6}$$

A lock-in amplifier multipliesthe measured signal A_{sig} to a reference signal A_{ref}. The inputs are given by

$$A_{\text{sig}} = V_{\text{sig}}\sin(\omega_{\text{sig}}t + \theta_{\text{sig}}) \qquad A_{\text{ref}} = V_{\text{ref}}\sin(\omega_{\text{ref}}t + \theta_{\text{ref}}) \tag{7}$$

Making use of Equation (6), multiplication of the two signals leads to

$$A_{\text{mult}} = \frac{1}{2}V_{\text{sig}}V_{\text{ref}}\Big[\cos\big((\omega_{\text{sig}} - \omega_{\text{ref}})t + \theta_{\text{sig}} - \theta_{\text{ref}}\big) - \cos\big((\omega_{\text{sig}} + \omega_{\text{ref}})t + \theta_{\text{sig}} + \theta_{\text{ref}}\big)\Big] \tag{8}$$

Next, A_{mult} is sent through a low pass filter, removing high frequency signals: only signals with $\omega_{\text{sig}} - \omega_{\text{ref}}$ smaller than the bandwidth of the filter can pass. In particular, the $\cos(\omega_{\text{sig}} + \omega_{\text{ref}})$ component cannot pass. A digital low pass filter can be thought of as averaging over small time steps such that high frequencies will cancel out.

For $\omega_{\text{sig}} = \omega_{\text{ref}}$, the output signal is given by

$$A_{\text{out}} = \frac{1}{2}V_{\text{sig}}V_{\text{ref}}\cos\big(\theta_{\text{sig}} - \theta_{\text{ref}}\big) \tag{9}$$

This is an DC signal unless $\theta_{\text{sig}} - \theta_{\text{ref}}$ changes over time. The process of multiplying and then filtering is known as **demodulation**, which in general means to filter out a desired

frequency signal from a high frequency carrier wave.
Any signal, like noise, whose frequency differs from the one of interest is strongly filtered out. However, it may lead to a low frequency output if $\omega_{\text{ref}} \approx \omega_{\text{noise}}$, meaning that it can just pass the low pass filter.

By taking the reference signal, splitting it up and shifting one of them by $\pi/2$, we can perform the procedure described above twice. Labelling the two outputs $A_X (= A_{\text{out}})$ and $A_Y (= A_{\text{shifted}})$, we obtain

$$A_{\text{X}}^2 = \frac{1}{4} V_{\text{sig}}^2 V_{\text{ref}}^2 \cos^2(\theta), \quad A_{\text{Y}}^2 = \frac{1}{4} V_{\text{sig}}^2 V_{\text{ref}}^2 \sin^2(\theta), \tag{10}$$

where $\theta \coloneqq \theta_{\text{sig}} - \theta_{\text{ref}}$ is the relative phase. The following important properties hold:

$$\sqrt{A_{\text{X}}^2 + A_{\text{Y}}^2} = \frac{1}{2} \left| V_{\text{sig}} V_{\text{ref}} \right| = \frac{1}{2} V_{\text{sig}} V_{\text{ref}} \tag{11}$$

$$\tan^{-1}\left(\frac{A_Y}{A_X}\right) = \theta \tag{12}$$

Where we used that V_{sig}, $V_{\text{ref}} > 0$ since they are amplitudes. A schematic of a lock-in amplifier can be seen in Figure 2.

We have now fully characterised the signal with respect to the reference signal. Equations (11) and (12) allow to represent it in phase space.

For this experiment, the lock-in amplifier used is the MFLI Lock-in Amplifier by Zurich Instruments, see [3].

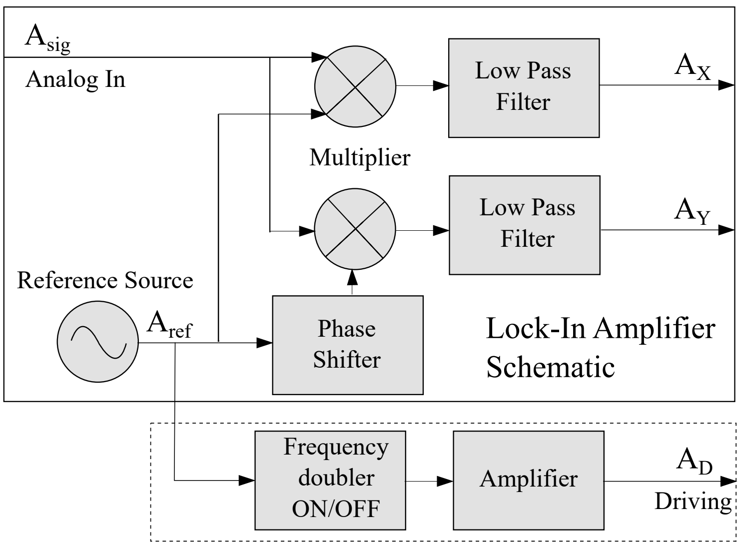

Figure 2: Schematic of a Lock-In Amplifier. A_X and A_Y are the x and y components, respectively, of the signal in phase space. Part of the reference signal can be frequency doubled and amplified if needed to serve as a driving force.

4 LabOne

The MFLI Lock-in Amplifier is controlled using the LabOne software, see [4]. LabOne comprises many important and useful features, some of which are outlined below.

i) The *Numeric* functionality, which gives the numerical value of the current signal.

ii) The *Plotter*, which plots the signal as a function of time. The time interval can be adjusted. One can plot simultaneously the demodulator amplitude R, X and Y, and the phase ϕ.
For long time intervals, the file size is big (multiple MB for an interval $[-60\,\mathrm{s}, 0\,\mathrm{s}]$).

iii) The *Sweeper*, where one variable, in this case the driving frequency, is varied, and the signal is recorded as a function of this variable. One can adjust the sweep-interval, step-size, waiting time (time, before a new data point is collected), and the number of data points collected per step. Choosing sensible settings for the waiting time and data points per step can improve signal to noise ratio, while the step-size is crucial for observing hysteresis behaviour.
The driving frequency sweep-interval can be swept *sequentially* (from low to high frequencies), in *reverse*, or in both directions, *bidirectional*.

iv) The *Scope*, which collects data over a range of frequencies and performs a Fast Fourier Transform (FFT). The data-collection frequency and frequency interval can be varied. The scope can be used to find natural frequencies (normal modes) of the system.

v) The *DAQ* functionality, which acquires and displays the signal and can be operated for single measurements as well as continuous ones.

LabOne can be used to control the internal reference signal of the MFLI Lock-in Amplifier as well as generating an output signal. The output signal is obtained by adjusting the reference signal. These adjustments include the amplification of the signal, the relative phase to the reference signal, as well as setting the ratio of the output and reference frequencies $\omega_{\mathrm{output}}/\omega_{\mathrm{ref}}$. For driving force measurements the ratio is set to 1, for parametric resonance measurements the ratio is set to 2.

Collected data can be saved directly from LabOne into `.txt` or `.csv` files.

5 Experiment

5.1 Set-Up

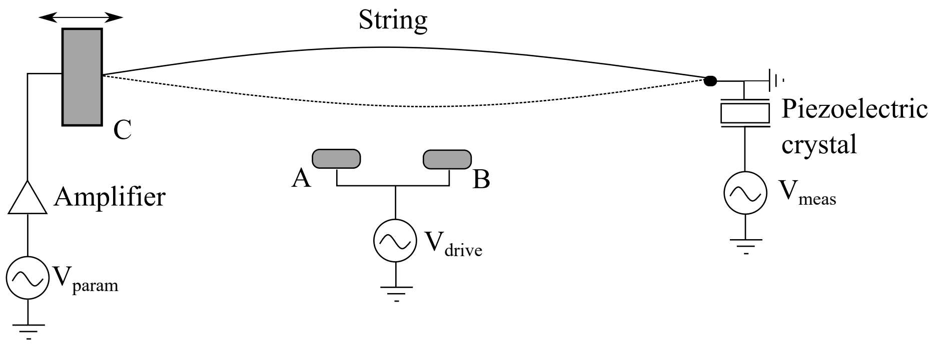

Figure 3: The set-up for driving force and parametric resonance measurements. V_{param} and V_{drive} are controlled by the lock-in amplifier. V_{meas} is the measured voltage, which is input into the lock-in amplifier. V_{param} is amplified by an external amplifier.

The oscillator considered in this experiment is a steel string, as depicted in Figure 3.
Driving Force. A magnet is attached to the string. By passing an AC current through either coil A or B, the string is excited. V_{drive} is output by the lock-in amplifier at the same frequency as the reference signal.
Parametric Modulation. To one clamp of the string, a magnet is attached. By applying a current to coil C, the magnet is moved, changing the length of the string. The natural frequency of the string is thus parametrically modulated. V_{param} is output by the lock-in amplifier at twice the frequency of the reference signal, and is amplified in an external amplifier.
The output signal of the MFLI Lock-in Amplifier must not exceed 1 V.

The coils A and B can serve as a detector, because the oscillating magnet induces a voltage across the coil. Alternatively, to improve the signal-to-noise ratio, a piezoelectric crystal is placed under one of the string clamps. The string oscillation compresses the crystal, thus creating a voltage with amplitude proportional to the oscillation amplitude. The signal V_{meas} is sent to the MFLI Lock-in Amplifier (signal port IN).

5.2 Measurements

The goal of this experiment is to observe nonlinear behaviour of the resonator, as well as the effects of parametric driving, in particular hysteresis behaviour.

- Start the MFLI Lock-in Amplifier and LabOne. Familiarise yourself with the interface.
- Estimate the natural frequency of the string by using the *Scope*.
  - → What are sensible settings for the frequency range and sampling rate?
  - → How can you excite a string oscillation? (Do **not** touch the string directly.)
  - → Which peaks are caused by environmental factors?
  - → Of the remaining peaks, which is the frequency around which measurements should be conducted?
- Set up the experiment for measurements with a driving force and no parametric modulation.
- Use the *Sweeper* to find the response of the system as a function of driving frequency for different driving voltages.
  - → Choose appropriate step size, waiting time, and sweep interval. To estimate these values, one can use the *Plotter*, which records in real time the Lock-in amplifier readouts, to observe the settling time after being excited of the oscillator.
  - → How does the shape of the response curves change for different driving voltages? Is it Lorentzian? Fit the data and extract Q.
  - → How does the resonance frequency change for different driving forces? Compare this to (3).
- Use the *Plotter* to measure the **ring-down** (decay of the signal from a steady oscillation) near resonance. Does the amplitude decay exponentially? Which parameter in (1) does the exponential decay correspond to (at small amplitudes)? Does the phase of the oscillation affect the ring-down?
- Set up the experiment for measurements with parametric modulation and no driving force.
- Measure the **ring-up** (settling of the oscillation from rest to a steady state) of the oscillation into both the degenerate phase states for parametric driving, using the *Plotter*. Then, measure the ring-down from the states. Can you fit additional parameters in (1)?
- Use the *Sweeper* and sweep over the modulation frequency for different modulation depths λ. *Note: The sweep should be done in the direction opposite to the bending of the resonance.*

→ Observe the effect of the sweep step size on hysteresis behaviour. Do you need to choose different step sizes for different modulation depths? How does the waiting time affect the results?

→ Find the shape of the Arnold's Tongue (instability region).

→ Estimate the instability threshold λ_{th}.

- Observe hysteresis behaviour by sweeping bidirectionally over the frequency interval.

For an additional point in the experiment, the following tasks can be performed.

- Study the Zurich Instruments API and write python scripts to control the measurements. This should, among other functions, contain the ability to perform and store a phase space measurement from the *DAQ* of the MFLI in a useable data format, and further read and plot the data in the X and Y quadratures.

- Any oscillator when subject to a finite temperature is subject to thermal noise. Measure the thermal noise of the string resonator using the X and Y quadratures in the *Plotter*.

- Apply an external force to the oscillator. Perform a noise measurement. What changes do you see? *(Hint: Look at the magnitudes of the quadratures as well as the shape!)*

- Research *noise squeezing*. A parametric pump leads to phase-dependent damping coefficients below the instability threshold λ_{th}. As a result, the oscillator prefers to be in one quadrature over the other. This leads to a "squeezing effect" that can be observed in the noise measurements.

- Use the resonator to observe squeezing effects at various parametric drives below λ_{th}. Comment on your observations.

- Drive above λ_{th} with no external force. Add external noise using the Arbitrary Waveform Generator. Investigate the quadrature plots.

References

[1] A. Leuch *et al.*, Phys. Rev. Lett. **117**, 21 (2016). *Parametric Symmetry Breaking in a Nonlinear Resonator.*

[2] M. C. Lifshitz and R. Cross, *Nonlinear Dynamics of Nanomechanical and Micromechanical Resonators* (Wiley-VCH, Weinheim, 2009), pp. 1 - 52.

[3] Zurich Instruments, MFLI Lock-in Amplifier, 30 October 2019, `zhinst.com/products/mfli`

[4] Zurich Instruments, LabOne, 30 October 2019, `zhinst.com/labone`